CARL STITZ

# COLLEGE ALGEBRA

CARL STITZ
JEFF ZEAGER

# COLLEGE ALGEBRA

# Table of Contents

**Preface** v

**0 Basic Algebra** 1
- 0.1 The Laws of Algebra . . . 1
- 0.2 Kinds of Numbers . . . 6
- 0.3 Exponents . . . 9
- 0.4 Absolute Value . . . 10
- 0.5 Solving Equations . . . 12
- 0.6 Exercises . . . 13
- 0.7 Answers . . . 14

**1 Coordinates** 15
- 1.1 The Cartesian Coordinate Plane . . . 15
  - 1.1.1 Distance in the Plane . . . 20
  - 1.1.2 Exercises . . . 25
  - 1.1.3 Answers . . . 27
- 1.2 Relations . . . 28
  - 1.2.1 Exercises . . . 32
  - 1.2.2 Answers . . . 34
- 1.3 Graphs of Equations . . . 36
  - 1.3.1 Exercises . . . 42
  - 1.3.2 Answers . . . 43
- 1.4 Three Interesting Curves . . . 49
  - 1.4.1 Circles . . . 49
  - 1.4.2 Parabolas . . . 53
  - 1.4.3 Ellipses . . . 61
  - 1.4.4 Exercises . . . 66
  - 1.4.5 Answers . . . 69

**2 Functions** 77
- 2.1 Introduction to Functions . . . 77
  - 2.1.1 Exercises . . . 83

2.1.2 Answers . . . 86
2.2 Function Notation . . . 87
2.2.1 Exercises . . . 93
2.2.2 Answers . . . 96
2.3 Function Arithmetic . . . 98
2.3.1 Exercises . . . 103
2.3.2 Answers . . . 105
2.4 Graphs of Functions . . . 106
2.4.1 General Function Behavior . . . 113
2.4.2 Exercises . . . 120
2.4.3 Answers . . . 123
2.5 Transformations . . . 127
2.5.1 Exercises . . . 149
2.5.2 Answers . . . 151

**3 Linear and Quadratic Functions** **157**
3.1 Linear Functions . . . 157
3.1.1 Exercises . . . 170
3.1.2 Answers . . . 173
3.2 Defining Functions (Word Problems) . . . 175
3.2.1 Exercises . . . 181
3.2.2 Answers . . . 181
3.3 Quadratic Functions . . . 182
3.3.1 Exercises . . . 192
3.3.2 Answers . . . 194
3.4 Inequalities . . . 198
3.4.1 Exercises . . . 211
3.4.2 Answers . . . 212

**4 Polynomial Functions** **215**
4.1 Graphs of Polynomials . . . 215
4.1.1 Exercises . . . 228
4.1.2 Answers . . . 231
4.2 The Factor Theorem and The Remainder Theorem . . . 235
4.2.1 Exercises . . . 244
4.2.2 Answers . . . 245

**5 Rational Functions** **247**
5.1 Introduction to Rational Functions . . . 247
5.1.1 Exercises . . . 259
5.1.2 Answers . . . 261
5.2 Graphs of Rational Functions . . . 263
5.2.1 Exercises . . . 276

5.2.2 Answers . . . 278
5.3 Rational Inequalities and Applications . . . 284
5.3.1 Exercises . . . 291
5.3.2 Answers . . . 293

**6 Further Topics in Functions 295**
6.1 Function Composition . . . 295
6.1.1 Exercises . . . 305
6.1.2 Answers . . . 307
6.2 Inverse Functions . . . 309
6.2.1 Exercises . . . 328
6.2.2 Answers . . . 329

**7 Exponential and Logarithmic Functions 331**
7.1 Introduction to Exponential and Logarithmic Functions . . . 331
7.1.1 Exercises . . . 345
7.1.2 Answers . . . 348
7.2 Properties of Logarithms . . . 350
7.2.1 Exercises . . . 359
7.2.2 Answers . . . 361
7.3 Exponential Equations and Inequalities . . . 362
7.3.1 Exercises . . . 371
7.3.2 Answers . . . 372
7.4 Logarithmic Equations and Inequalities . . . 373
7.4.1 Exercises . . . 381
7.4.2 Answers . . . 383
7.5 Applications of Exponential and Logarithmic Functions . . . 384
7.5.1 Applications of Exponential Functions . . . 384
7.5.2 Applications of Logarithms . . . 393
7.5.3 Exercises . . . 395
7.5.4 Answers . . . 397

**8 Systems of Equations 399**
8.1 Systems of Linear Equations: Gaussian Elimination . . . 399
8.1.1 Exercises . . . 414
8.1.2 Answers . . . 416
8.2 Systems of Linear Equations: Augmented Matrices* . . . 419
8.2.1 Exercises . . . 428
8.2.2 Answers . . . 431
8.3 Determinants and Cramer's Rule* . . . 432
8.3.1 Definition and Properties of the Determinant . . . 432
8.3.2 Cramer's Rule . . . 434
8.3.3 Exercises . . . 437

8.3.4 Answers . . . 438
8.4 Systems of Non-Linear Equations and Inequalities . . . 438
8.4.1 Exercises . . . 448
8.4.2 Answers . . . 451

**9 Sequences and Series 455**
9.1 Sequences . . . 455
9.1.1 Exercises . . . 462
9.1.2 Answers . . . 463
9.2 Series and Summation Notation . . . 463
9.2.1 Exercises . . . 469
9.2.2 Answers . . . 469
9.3 IRAs and Mortgages . . . 470
9.3.1 Exercises . . . 473
9.3.2 Answers . . . 473
9.4 Infinite sums and Repeating Decimals* . . . 473
9.4.1 Exercises . . . 476

**10 Complex Numbers and the Fundamental Theorem of Algebra 477**
10.1 Complex Numbers . . . 477
10.2 The Fundamental Theorem of Algebra . . . 480
10.2.1 Exercises . . . 487
10.2.2 Answers . . . 488

**A The Laws of Algebra Proved 491**
A.1 The Laws of Algebra . . . 491
A.2 The Analogy between Addition and Multiplication . . . 491
A.3 Consequences of the Distributive Law . . . 496

# Preface

This book is a modified version of the Open Source Precalculus Project initiated by Carl Stitz and Jeff Seager. The original version is available at

http://www.stitz-zeager.com/Free_College_Algebra_Book_Download.html.

As indicated on that website you may go to

http://www.lulu.com/product/paperback/college-algebra/11396948

to order a low-cost, royalty free printed version of the book from lulu.com. Neither author receives royalties from lulu.com, and, in most cases, it is far cheaper to purchase the printed version from lulu than to print out the entire book at home.

The version you are viewing was modified by Joel Robbin and Mike Schroeder for use in Math 112 at the University of Wisconsin Madison. A companion workbook for the course is being published by Kendall Hunt Publishing Co. 4050 Westmark Drive, Dubuque, IA 52002. Neither Joel Robbin nor Mike Schroeder nor anyone else at the University of Wisconsin receives any royalties from sales of the workbook to UW students.

The original version of this book contains the following acknowledgements:

> The authors are indebted to the many people who support this project. From Lakeland Community College, we wish to thank the following people: Bill Previts, who not only class tested the book but added an extraordinary amount of exercises to it; Rich Basich and Ivana Gorgievska, who class tested and promoted the book; Don Anthan and Ken White, who designed the electric circuit applications used in the text; Gwen Sevits, Assistant Bookstore Manager, for her patience and her efforts to get the book to the students in an efficient and economical fashion; Jessica Novak, Marketing and Communication Specialist, for her efforts to promote the book; Corrie Bergeron, Instructional Designer, for his enthusiasm and support of the text and accompanying YouTube videos; Dr. Fred Law, Provost, and the Board of Trustees of Lakeland Community College for their strong support and deep commitment to the project. From Lorain County Community College, we wish to thank: Irina Lomonosov for class testing the book and generating accompanying PowerPoint slides; Jorge Gerszonowicz, Kathryn Arocho, Heather Bubnick, and Florin Muscutariu for their unwavering support of the project; Drs. Wendy Marley and Marcia Ballinger, Lorain CCC, for the Lorain CCC

enrollment data used in the text. We would also like to extend a special thanks to Chancellor Eric Fingerhut and the Ohio Board of Regents for their support and promotion of the project. Last, but certainly not least, we wish to thank Dimitri Moonen, our dear friend from across the Atlantic, who took the time each week to e-mail us typos and other corrections.

# Chapter 0

# Basic Algebra

## 0.1 The Laws of Algebra

**Terminology and Notation.** In this section we review the notations used in algebra. Some are peculiar to this book. For example the notation $A := B$ indicates that the equality holds *by definition* of the notations involved. Two other notations which will become important when we solve equations are $\implies$ and $\iff$. The notation $P \implies Q$ means that $P$ *implies* $Q$ i.e. *"If $P$, then $Q$"*. For example, $x = 2 \implies x^2 = 4$. (Note however that the converse statement $x^2 = 4 \implies x = 2$ is not always true since it might be that $x = -2$.) The notation $P \iff Q$ means $P \implies Q$ and $Q \implies P$, i.e. *"$P$ if and only if $Q$"*. For example $3x - 6 = 0 \iff x = 2$. The notations $\implies$ and $\iff$ are explained more carefully in Section 0.5 below.

**Implicit Multiplication.** In mathematics the absence of an operation symbol usually indicates multiplication: $ab$ mean $a \times b$. Sometimes a dot is used to indicate multiplication and in computer languages an asterisk is often used.

$$ab := a \cdot b := a * b := a \times b$$

**Order of operations.** Parentheses are used to indicate the order of doing the operations: in evaluating an expression with parentheses the innermost matching pairs are evaluated first as in

$$((1+2)^2+5)^2 = (3^2+5)^2 = (9+5)^2 = 14^2 = 196.$$

There are conventions which allow us not to write the parentheses. For example, multiplication is done before addition

$$ab + c \qquad \text{means } (ab) + c \text{ and not } a(b+c),$$

and powers are done before multiplication:

$$ab^2c \qquad \text{means } a(b^2)c \text{ and not } (ab)^2c.$$

In the absence of other rules and parentheses, the left most operations are done first.

$$a - b - c \qquad \text{means } (a-b) - c \text{ and not } a - (b-c).$$

The long fraction line indicates that the division is done last:

$$\frac{a+b}{c} \qquad \text{means } (a+b)/c \text{ and not } a+(b/c).$$

In writing fractions the length of the fraction line indicates which fraction is evaluated first:

$$\frac{a}{\frac{b}{c}} \qquad \text{means } a/(b/c) \text{ and not } (a/b)/c,$$

$$\frac{\frac{a}{b}}{c} \qquad \text{means } (a/b)/c \text{ and not } a/(b/c).$$

The length of the horizontal line in the radical sign indicates the order of evaluation:

$$\sqrt{a+b} \qquad \text{means } \sqrt{(a+b)} \text{ and not } (\sqrt{a})+b.$$

$$\sqrt{a}+b \qquad \text{means } (\sqrt{a})+b \text{ and not } \sqrt{(a+b)}.$$

**The Laws of Algebra.** There are four fundamental operations which can be performed on numbers.

1. Addition. The **sum** of $a$ and $b$ is denoted $a+b$.
2. Multiplication. The **product** of $a$ and $b$ is denoted $ab$.
3. Reversing the sign. The **negative** of $a$ is denoted $-a$.
4. Inverting. The **reciprocal** of $a$ (for $a \neq 0$) is denoted by $a^{-1}$ or by $\dfrac{1}{a}$.

These operations satisfy the following laws.

| | | |
|---|---|---|
| Associative | $a+(b+c)=(a+b)+c$ | $a(bc)=(ab)c$ |
| Commutative | $a+(b+c)=(a+b)+c$ | $a(bc)=(ab)c$ |
| Identity | $a+0=0+a=a$ | $a\cdot 1=1\cdot a=a$ |
| Inverse | $a+(-a)=(-a)+a=0$ | $a\cdot a^{-1}=a^{-1}\cdot a=1$ |
| Distributive | $a(b+c)=ab+ac$ | $(a+b)c=ac+bc$ |

The operations of **subtraction** and **division** are then defined by

$$a-b:=a+(-b) \qquad\qquad a\div b:=\frac{a}{b}:=a\cdot b^{-1}=a\cdot\frac{1}{b}.$$

All the rules of calculation that you learned in elementary school follow from the above fundamental laws. In particular, the Commutative and Associative Laws say that you can add a bunch of numbers in any order and similarly you can multiply a bunch of numbers in any order. For example,

$$(A+B)+(C+D)=(A+C)+(B+D), \qquad (A\cdot B)\cdot(C\cdot D)=(A\cdot C)\cdot(B\cdot D).$$

Because both addition and multiplication satisfy the commutative, associative, identity, and inverse laws, there are other analogies:

| | | |
|---|---|---|
| **(i)** | $-(-a)=a$ | $(a^{-1})^{-1}=a$ |
| **(ii)** | $-(a+b)=-a-b$ | $(ab)^{-1}=a^{-1}b^{-1}$ |
| **(iii)** | $-(a-b)=b-a$ | $\left(\frac{a}{b}\right)^{-1}=\frac{b}{a}$ |
| **(iv)** | $(a-b)+(c-d)=(a+c)-(b+d)$ | $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ |
| **(v)** | $a-b=(a+c)-(b+c)$ | $\frac{a}{b}=\frac{ac}{bc}$ |
| **(vi)** | $(a-b)-(c-d)=(a-b)+(d-c)$ | $\frac{a/b}{c/d}=\frac{a}{b}\cdot\frac{d}{c}$ |

These identities[1] are proved in Appendix A.2 Here are some further identities which are proved using the distributive law.

[1] An **identity** is an equation which is true for all values of the variables which appear in it.

**(i)** $a \cdot 0 = 0$ **(ii)** $-a = (-1)a$

**(iii)** $a(-b) = -ab$ **(iv)** $(-a)(-b) = ab$

**(v)** $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + cb}{bd}$ **(vi)** $(a + b)(c + d) = ab + ad + bc + bd$

**(vii)** $(a + b)^2 = a^2 + 2ab + b^2$ **(viii)** $(a + b)(a - b) = a^2 - b^2$

These proved in Appendix A.3.

An important consequence of the fact that $a \cdot 0 = 0 \cdot a = 0$ is the following **Zero-Product Property**. It is used to solve equations.

Zero-Product Property

$pq = 0 \iff p = 0$ or $q = 0$ (or both).

Proof: If $p = 0$ (or $q = 0$) then $pq = 0$. Conversely, if $p \neq 0$, then $q = p^{-1}pq = p^{-1}0 = 0$. □

DEFINITION 0.1. For a natural number $n$ and any number $a$ the $n$th **power** of $a$ is

$$a^n := \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

The zeroth power is

$$a^0 := 1$$

and negative powers are defined by

$$a^{-n} := \frac{1}{a^n}.$$

**The Laws of Exponents.** The following laws are easy to understand when $m$ and $n$ are integers. In Theorem 0.1 below we will learn that these laws also hold whenever $a$ and $b$ are positive real numbers and $m$ and $n$ are *any* real numbers, not just integers.

| | | | |
|---|---|---|---|
| **(i)** | $a^m a^n = a^{m+n}$ | e.g. | $a^2a^3 = (aa)(aaa) = a^5$ |
| **(ii)** | $(a^m)^n = a^{mn}$ | e.g. | $(a^2)^3 = (aa)(aa)(aa) = a^6$ |
| **(iii)** | $\dfrac{a^m}{a^n} = a^{m-n}$ | e.g. | $\dfrac{a^2}{a^5} = a^{-3} = \dfrac{1}{a^3}$ |
| **(iv)** | $(ab)^m = a^m b^m$ | e.g. | $(ab)^2 = (ab)(ab) = (aa)(bb) = a^2b^2$ |
| **(v)** | $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ | e.g. | $\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b}\cdot\dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$ |

Example 0.1.1. Simplify $\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^{-1}$

Solution.

$$\left(\frac{1}{a}+\frac{1}{b}\right)^{-1} = \frac{1}{\frac{1}{a}+\frac{1}{b}} = \frac{ab}{\left(\frac{1}{a}+\frac{1}{b}\right)ab} = \frac{ab}{\frac{ab}{a}+\frac{ab}{b}} = \frac{ab}{b+a}.$$

**Three Column Calculations.** An algebraic calculation often involves substituting expressions for letters in general laws. To avoid making mistakes it is advisable to arrange the computation neatly and use equal signs between quantities which you assert are equal. When you check your work, ask yourself at each step what general principle you used and how you substituted into that general expression. The following **three column calculation** illustrates this technique.[2]

| step | by | with |
|---|---|---|
| $\dfrac{1}{3}+\dfrac{2}{x} = \dfrac{x}{3x}+\dfrac{2}{x}$ | $\dfrac{A}{B} = \dfrac{AC}{CB}$ | $A=1,\ B=3,\ C=x$ |
| $= \dfrac{x}{3x}+\dfrac{6}{3x}$ | $\dfrac{A}{B} = \dfrac{CA}{CB}$ | $A=2,\ B=x,\ C=3$ |
| $= \dfrac{x+6}{3x}$ | $\dfrac{A}{C}+\dfrac{B}{C} = \dfrac{A+B}{C}$ | $A=x,\ B=6,\ C=3x$ |

**When in Doubt.** If you are in doubt as to whether some general equation is true you can plug in numbers: if the two sides of the equation are not equal the general equation is false.[3] Thus, in general,

$$(A+B)^2 \neq A^2+B^2$$

[2] We *don't* expect this level of detail on exams; we *do* expect you to do calculations like this without making mistakes.

[3] But maybe not conversely!

because when $A = 2$ and $B = 3$ we have $(A + B)^2 = (2 + 3)^2 = 5^2 = 25$ but $A^2 + B^2 = 2^2 + 3^2 = 4 + 9 = 13$ and $25 \neq 13$. The correct general law is

$$(A + B)^2 = A^2 + 2AB + B^2$$

(when $A = 2$ and $B = 3$ $A^2 + 2AB + B^2 = 4 + 12 + 9 = 25 = 5^2 = (2 + 3)^2$) and this shows that $(A + B)^2 = A^2 + B^2 \implies 2AB = 0$ so that $(A + B)^2 = A^2 + B^2$ only when $A = 0$ or $B = 0$ (or both).

## 0.2 KINDS OF NUMBERS

We distinguish the following different kinds of numbers.

- The **natural numbers** are $1, 2, 3 \ldots$.
- The **integers** are $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$.
- The **rational numbers** are ratios of integers like $3/2$, $14/99$, $-1/2$.
- The **real numbers** are numbers which have an infinite decimal expansion like

$$\frac{3}{2} = 1.5000\ldots, \quad \frac{14}{99} = 0.141414\ldots, \quad \sqrt{2} = 1.4142135623730951\ldots.$$

- The **complex numbers** are those numbers of form $z = x+iy$ where $x$ and $y$ are real numbers and $i$ is a special new number called the **imaginary unit** which has the property that

$$i^2 = -1;$$

Every integer is a rational number (because $n = n/1$), every rational number is a real number (see Remark 0.2 below), and every real number is a complex number (because $x = x + 0i$). A real number which is not rational is called **irrational**.

**New Numbers - New Solutions.** Each kind of number enables us to solve equations that the previous kind couldn't solve:

- The solution of the equation $x + 5 = 3$ is $x = -2$ which is an integer but not a natural number.
- The solution of the equation $5x = 3$ is $x = \frac{3}{5}$ which is a rational number but not an integer.
- The equation $x^2 = 2$ has two solutions $x = \sqrt{2}$. The number $\sqrt{2}$ is a real number but not a rational number.
- The equation $x^2 = 4$ has two real solutions $x = \pm 2$ but the equation $z^2 = -4$ has no real solutions because the square of a nonzero real number is always positive. However it does have two complex solutions, namely $z = \pm 2i$.

We will not use complex numbers until Chapter 10 but may refer to them implicitly as in

*The equation $x^2 = -4$ has no (real) solution.*

**Rational Numbers - Repeating Decimals.** It will be proved in Theorem 9.3 that a real number is rational if and only if its decimal expansion eventually repeats periodically forever as in the following examples:

$$\frac{1}{3} = 0.3333\ldots, \qquad \frac{17}{6} = 2.83333\ldots,$$

$$\frac{7}{4} = 1.250000\ldots, \qquad \frac{22}{7} = 3.142857\,142857\,142857\ldots.$$

Unless the decimal expansion of a real number is eventually zero, as in $\frac{1}{2} = 0.5000\ldots$, any finite part of the decimal expansion is close to, but not exactly equal to, the real number. For example 1.414 is close to the square root of two but not exactly equal:

$$(1.414)^2 = 1.999396 \neq 2, \qquad (\sqrt{2})^2 = 2.$$

If we compute the square root to more decimal places we get a better approximation, but it still isn't exactly correct:

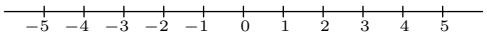
$$(1.4142135623730951)^2 = 2.00000000000000014481069235364401.$$

**The square root of 2 is Irrational.** Here is a proof that $\sqrt{2}$ is irrational. If it were rational there would be integers $m$ and $n$ with

$$\left(\frac{m}{n}\right)^2 = 2.$$

By canceling common factors we may assume that $m$ and $n$ have no common factors and hence that they are not both even. Now $m^2 = 2n^2$ so $m^2$ is even so $m$ is even, say $m = 2p$. Then $4p^2 = (2p)^2 = m^2 = 2n^2$ so $2p^2 = n^2$ so $n^2$ is even so $n$ is even. This contradicts the fact $m$ and $n$ are not both even.

**The Number Line.** The choice of two points (representing 0 and 1) on a line determines a correspondence between the points of the line and the real numbers as indicated in the following picture.




The correspondence is called a **coordinate system** on the line. The line is called a **number line**. When the point $A$ corresponds to the number $a$ we say that the number $a$ is the **coordinate** of the point $A$. The **positive numbers** are the real numbers on the same side of 0 as 1 and the **negative numbers** are on the other side. We usually draw the number line as above so that it is

horizontal and 1 is to the right of 0. We write say $a$ is **less than** $b$ and write $a < b$ $b$ is to the right of $a$, i.e. when $b - a$ is positive. it is equivalent to say that $b$ is **greater than** $a$ or $a$ to the left of $b$ and to write $b > a$. The notation $a \leq b$ means that $a$ is less than or equal to $b$ i.e. either $a < b$ or else $a = b$. Similarly, $b \geq a$ means that $b$ is greater than or equal to $a$ i.e. either $b > a$ or else $b = a$. Thus when $a < b$, a number [4]

$$c \text{ is between } a \text{ and } b \iff a < c < b.$$

Sometimes we insert the word **strictly** for emphasis: $a$ is strictly less than $b$ means that $a < b$ (not just $a \leq b$).

**Order.** The **order relation** just described is characterized by the following.

**(Trichotomy)** Every real number is either positive, negative, or zero (and no number satisfies two of these conditions).

**(Sum)** The sum of two positive numbers is positive.

**(Product)** The product of two positive numbers is positive.

This characterization together with the notation explained in the previous paragraph implies the following:

**(i)** Either $a < b$, $a = b$, or $a > b$.

**(ii)** If $a < b$ and $b < c$, then $a < c$.

**(iii)** If $a < b$, then $a + c < b + c$.

**(iv)** If $a < b$ and $c > 0$, then $ac < bc$.

**(v)** If $a < b$ and $c < 0$, then $ac > bc$.

**(vi)** If $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

**Interval Notation**. The **open interval** $(a, b)$ is the set of all real numbers $x$ such that $a < x < b$, and the **closed interval** $[a, b]$ is the set of all real numbers $x$ such that $a \leq x \leq b$. Thus

$$x \text{ is in the set } (a, b) \iff a < x < b$$

and

$$x \text{ is in the set } [a, b] \iff a \leq x \leq b.$$

[4] The notation $\iff$ is an abbreviation for "if and only if".

These notations are extended to include **half open intervals** and **unbounded intervals** as in

$$\begin{aligned} x \text{ is in the set } (a,b] &\iff a < x \le b, \\ x \text{ is in the set } (a,\infty) &\iff a < x, \\ x \text{ is in the set } (-\infty,a] &\iff x \le a, \qquad \text{etc.} \end{aligned}$$

The **union** symbol $\cup$ is used to denote a set consisting of more than one interval as in

$$x \text{ is in the set } (a,b) \cup (c,\infty) \iff \text{either } a < x < b \text{ or else } c < x.$$

The symbol $\infty$ is pronounced **infinity** and is used to indicate that an interval is unbounded. It is not a number so we never write $(c,\infty]$.

EXAMPLE 0.2.1. Which is bigger: $\pi$ or $\sqrt{10}$? (Don't use a calculator.)
SOLUTION. $\pi = 3.14\ldots < 3.15$. and

$$3.15^2 = (3+0.15)^2 = 3^2 + 2\times 3\times 0.15 + 0.15^2 = 9 + 0.90 + 0.0225 = 9.9225 < 10$$

so $\pi < 3.15 < \sqrt{10}$.

## 0.3 EXPONENTS

The proof of the following theorem requires a more careful definition of the set of real numbers than we have given and is best left for more advanced courses.

THEOREM 0.1. Suppose that $a$ is a positive real number. Then there is one and only one way to define $a^x$ for all real numbers $x$ such that

**(i)** $a^{x+y} = a^x \cdot a^y, \quad a^0 = 1, \quad a^1 = a, \quad 1^x = 1.$

**(ii)** If $a > 1$ and $x < y$ then $a^x < a^y$.

**(iii)** If $a < 1$ and $x < y$ then $a^x > a^y$.

With this definition, the laws of exponents in Paragraph 0.1 continue to hold when $a$ and $b$ are positive real numbers and $m$ and $n$ are arbitrary real numbers. The number $a^x$ is positive (when $a$ is positive) regardless of the sign of $x$.

In particular by property (v) in Paragraph 0.1 we have $(a^x)^y = a^{xy}$ so $(a^{m/n})^n = a^m$ and $(a^m)^{1/n} = a^{m/n}$. Hence for positive numbers $a$ and $b$ we have

$$b = a^{m/n} \iff b^n = a^m.$$

When $m = 1$ and $n$ is a natural number the number $a^{1/n}$ is called the $n$th **root** (**square root** if $n = 2$ and **cube root** if $n = 3$) and is sometimes denoted

$$\sqrt[n]{a} := a^{1/n}.$$

When $n$ is absent, $n = 2$ is understood:

$$\sqrt{a} := a^{1/2}.$$

**$n$th roots** A number $b$ is said to be an **$n$th root** of $a$ iff $b^n = a$. When $n$ is odd, every real number $a$ has exactly one (real) $n$th root and this is denoted by $\sqrt[n]{a}$. When $n$ is even, a positive real number $a$ has two (real) $n$th roots (and $\sqrt[n]{a}$ denotes the one which is positive) but a negative number has no real $n$th roots. (In trigonometry it is proved that every nonzero complex number has exactly $n$ distinct complex $n$th roots.)

The equation $b^2 = 9$ has two solutions, namely $b = 3$ and $b = -3$ and each is "a" square root of 9 but only $b = 3$ is "the" square root of 9. However $-2$ is the (only) real cube root of $-8$ because $(-2)^3 = -8$. The number $-9$ has no real square root (because $b^2 = (-b)^2 > 0$ if $b \neq 0$) but does have two complex square roots (because $(3i)^2 = (-3i)^2 = -9$). For most of this book[5] we only use real numbers and we say that

$$\sqrt{a} \text{ is undefined when } a < 0$$

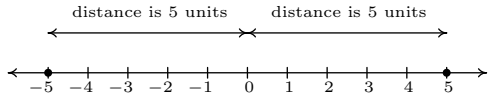
and that

*you can't take the square root of a negative number.*

Also $\sqrt{a}$ always denotes the nonnegative square root: thus $(-3)^2 = 3^2 = 9$ but $\sqrt{9} = 3$ and $\sqrt{9} \neq -3$.

## 0.4 Absolute Value

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number $x$. You may have been taught that $|x|$ is the distance from the real number $x$ to the 0 on the number. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line.




Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts

[5] More precisely until Chapter 10

while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

Definition 0.2. The **absolute value** of a real number $x$, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

In Definition 0.2, we define $|x|$ using a piecewise-defined function. (See page 93 in Section 2.2.) To check that this definition agrees with what we previously understood as absolute value, note that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$. This is one of the times when it's best to interpret the expression '$-x$' as 'the opposite of $x$' as opposed to 'negative $x$.' Before we embark on studying absolute value functions, we remind ourselves of the properties of absolute value.

Theorem 0.2. **Properties of Absolute Value:** Let $a$, $b$, and $x$ be real numbers and let $n$ be an integer.[a] Then

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever $a^n$ is defined
- **Quotient Rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided $b \neq 0$
- **The Triangle Inequality:** $|a + b| \leq |a| + |b|$
- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.

[a] Recall that this means $n = 0, \pm 1, \pm 2, \ldots$.

The proof of the Product and Quotient Rules in Theorem 0.2 boils down to checking four cases: when both $a$ and $b$ are positive; when they are both negative; when one is positive and the other is negative; when one or both are zero. For example, suppose we wish to show $|ab| = |a||b|$. We need to show this equation is true for all real numbers $a$ and $b$. If $a$ and $b$ are both positive, then so is $ab$. Hence, $|a| = a$, $|b| = b$, and $|ab| = ab$. Hence, the equation $|ab| = |a||b|$ is the same as

$ab = ab$ which is true. If both $a$ and $b$ are negative, then $ab$ is positive. Hence, $|a| = -a$, $|b| = -b$, and $|ab| = ab$. The equation $|ab| = |a||b|$ becomes $ab = (-a)(-b)$, which is true. Suppose $a$ is positive and $b$ is negative. Then $ab$ is negative, and we have $|ab| = -ab$, $|a| = a$ and $|b| = -b$. The equation $|ab| = |a||b|$ reduces to $-ab = a(-b)$ which is true. A symmetric argument shows the equation $|ab| = |a||b|$ holds when $a$ is negative and $b$ is positive. Finally, if either $a$ or $b$ (or both) are zero, then both sides of $|ab| = |a||b|$ are zero, and so the equation holds in this case, too. All of this rhetoric has shown that the equation $|ab| = |a||b|$ holds true in all cases. The proof of the Quotient Rule is very similar, with the exception that $b \neq 0$. The Power Rule can be shown by repeated application of the Product Rule. The last three properties can be proved using Definition 0.2 and by looking at the cases when $x \geq 0$, in which case $|x| = x$, or when $x < 0$, in which case $|x| = -x$. For example, if $c > 0$, and $|x| = c$, then if $x \geq 0$, we have $x = |x| = c$. If, on the other hand, $x < 0$, then $-x = |x| = c$, so $x = -c$. The remaining properties are proved similarly and are left as exercises.

## 0.5 SOLVING EQUATIONS

DEFINITION 0.3. A number $a$ is called a **solution** of an equation containing the variable $x$ if the equation becomes a true statement when $a$ is substituted for $x$. A solution of an equation is sometimes also called a **root** of the equation. Two equations are said to be **equivalent** iff they have exactly the same solutions. We will sometimes use the symbol $\iff$ to indicate that two equations are equivalent.

Usually two equations are equivalent because one can be obtained from the other by performing an operation to both sides of the equation which can be reversed by another operation of the same kind. For example, the equations $3x + 7 = 13$ and $x = 2$ are equivalent because

$$\begin{aligned} 3x + 7 = 13 &\iff 3x = 6 \text{ (subtract 7 from both sides)}, \\ &\iff x = 2 \text{ (divide both sides by 3)}. \end{aligned}$$

The reasoning is reversible: we can go from $x = 2$ to $3x = 6$ by multiplying both sides by by 3 and from $3x = 6$ to $3x + 7 = 13$ by adding 7 to both sides.

We use the symbol $\implies$ when we want to assert that one equation **implies** another but do not want to assert the converse. The guiding principal here is

> *If an equation $E'$ results from an equation $E$ by performing the same operation to both sides, then $E \implies E'$, i.e. every solution of $E$ is a solution of $E'$.*

If the operation is not "reversible" as explained above, there is the possibility that the set of solutions gets bigger in which case the new solutions are called **extraneous solutions**. (They do not solve the original equation.) The simplest example of how an extraneous solution can arise is

$$x = 3 \implies x^2 = 9 \qquad \text{(square both sides)}$$

but the operation of squaring both sides is not reversible: it is incorrect to conclude that $x^2 = 9$ implies that $x = 3$. What *is* correct is that $x^2 = 9 \iff x = \pm 3$, i.e. *either* $x = 3$ *or else* $x = -3$. When solving an equation you may use operations which are not reversible provided that you

*Always check your answer!*

In addition to catching mistakes, this will show you which – if any – of the solutions you found are extraneous.

Here are two ways in which extraneous solutions can arise:

**(i)** Squaring both sides of an equation.

**(ii)** Multiplying both sides of of an equation by a quantity not known to be nonzero.

As an example of (i) consider the equation

$$\sqrt{10-x} = -x - 2.$$

Squaring both sides gives the quadratic equation $10 - x = x^2 + 4x + 4$ which has two solutions $x = -6$ and $x = 1$. Now $\sqrt{10-(-6)} = -(-6) - 2$ but $\sqrt{10-1} \neq -1 - 2$ (Remember that $\sqrt{}$ means the positive square root.) Thus $x = -6$ is the only solution of the original equation and $x = 1$ is an extraneous solution.

As an example of (ii) consider

$$\frac{1}{x-1} = 2 + \frac{1}{x-1}.$$

This equation has no solution: if it did we would subtract $(x-1)^{-1}$ from both sides and deduce that $0 = 2$ which is false. But if we multiply both sides by $x - 1$ we get $1 = 2(x-1) + 1$ which has the (extraneous) solution $x = 1$.

## 0.6 Exercises

1. True or false?

   **(i)** $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$ ?

   **(ii)** $\dfrac{c}{a+b} = \dfrac{c}{a} + \dfrac{c}{b}$ ?

   **(iii)** $(a+b)/c = (a/c) + (b/c)$ ?

   **(iv)** $c/(a+b) = (c/a) + (c/b)$ ?

   **(v)** $\dfrac{a+b}{c+d} = \dfrac{a}{c} + \dfrac{b}{d}$ ?

   **(vi)** $\dfrac{a \cdot b}{c \cdot d} = \dfrac{a}{c} \cdot \dfrac{b}{d}$ ?

2. Factor:

   (a) $2x^3 - 12x^2 + 6x =$

   (b) $18a^4b^2 - 30a^3b^3 =$

   (c) $x^2 - a^2 =$

   (d) $4x^2 - \dfrac{1}{9} =$

3. Simplify:

   (a) $\dfrac{x^3-9x}{x^2+6x+9}$

   (b) $\dfrac{x^2+2x-3}{x^2+4x+4}\cdot\dfrac{x^2-4}{x^2+4x-5}$

   (c) $\dfrac{\left(\dfrac{1}{a}-2\right)}{\left(\frac{1}{a^2}-4\right)}$

   (d) $\dfrac{\left(\dfrac{1}{ab}+\dfrac{2}{ab^2}\right)}{\left(\frac{3}{a^3b}-\frac{4}{ab}\right)}$

4. Use the properties of exponents to rewrite the given expression as a simple fraction with only positive exponents.

   (a) $\left(\dfrac{k^{-4}s^2}{k^9s^{-7}}\right)^9$

5. Reduce the fraction $\dfrac{s+t}{sx^4+tx^4}$ to lowest terms.

## 0.7 ANSWERS

1. **(i)** True.

   **(ii)** False.: $\dfrac{1}{2+3}\neq\dfrac{1}{2}+\dfrac{1}{3}$

   **(iii)** True: $1/A=\dfrac{1}{A}$.

   **(iv)** False.

   **(v)** False: $\dfrac{2+3}{4+5}=\dfrac{5}{9},\quad \dfrac{2}{3}+\dfrac{4}{5}=\dfrac{22}{15}$.

   **(vi)** True.

2.

   (a) $2x^3-12x^2+6x=l2x(x^2-6x+3)$

   (b) $18a^4b^2-30a^3b^3=\ 6a^3b^2(3a-5b)$

   (c) $x^2-a^2=(x-a)(x+a)$

   (d) $4x^2-\dfrac{1}{9}=\left(2x-\dfrac{1}{3}\right)\left(2x+\dfrac{1}{3}\right)$

3.

   (a) $\dfrac{x^3-9x}{x^2+6x+9}=\dfrac{x(x-3)}{x+3}$

   (b) $\dfrac{x^2+2x-3}{x^2+4x+4}\cdot\dfrac{x^2-4}{x^2+4x-5}=\dfrac{(x+3)(x-2)}{(x+2)(x+5)}$

   (c) $\dfrac{\left(\dfrac{1}{a}-2\right)}{\left(\dfrac{1}{a^2}-4\right)}=\dfrac{a}{1+2a}$

   (d) $\dfrac{\left(\dfrac{1}{ab}+\dfrac{2}{ab^2}\right)}{\left(\dfrac{3}{a^3b}-\dfrac{4}{ab}\right)}=\dfrac{a^2(b+2)}{b(3-4a^2)}$

4. (a) $\left(\dfrac{k^{-4}s^2}{k^9s^{-7}}\right)^9=\left(\dfrac{s^9}{k^{13}}\right)^9=\dfrac{s^{81}}{k^{117}}$.

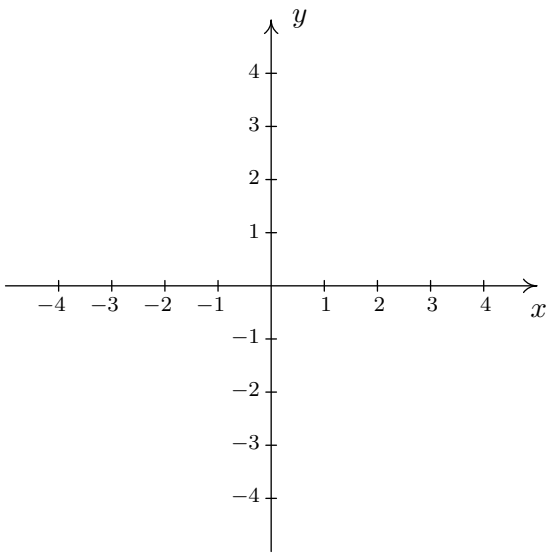
5. $\dfrac{s+t}{sx^4+tx^4}=\dfrac{s+t}{x^4(s+t)}=\dfrac{1}{x^4}$.

# Chapter 1

# Coordinates

## 1.1 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Algebra, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[1] Imagine two real number lines crossing at a right angle at 0 as below.

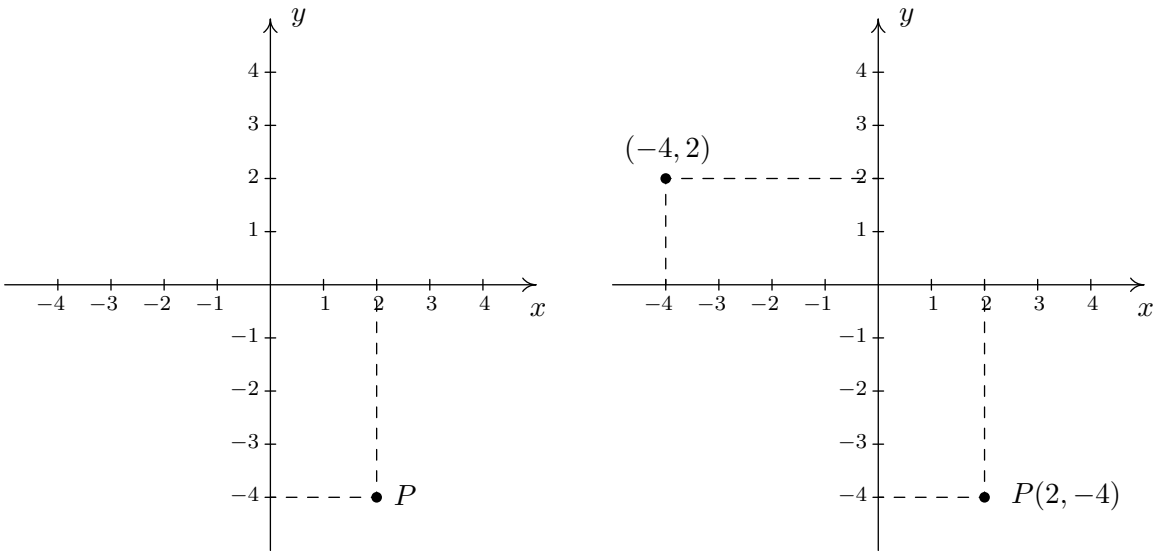



The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**.[2] As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the position of points off of the number lines as well as points on the lines themselves.

[1] So named in honor of René Descartes.

[2] The labels can vary depending on the context of application.

For example, consider the point $P$ below on the left. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the $x$-axis to $P$ and extending a horizontal line from the $y$-axis to $P$. We then describe the point $P$ using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**.[3] Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point $P$. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach $P$ from the origin by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important − if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.

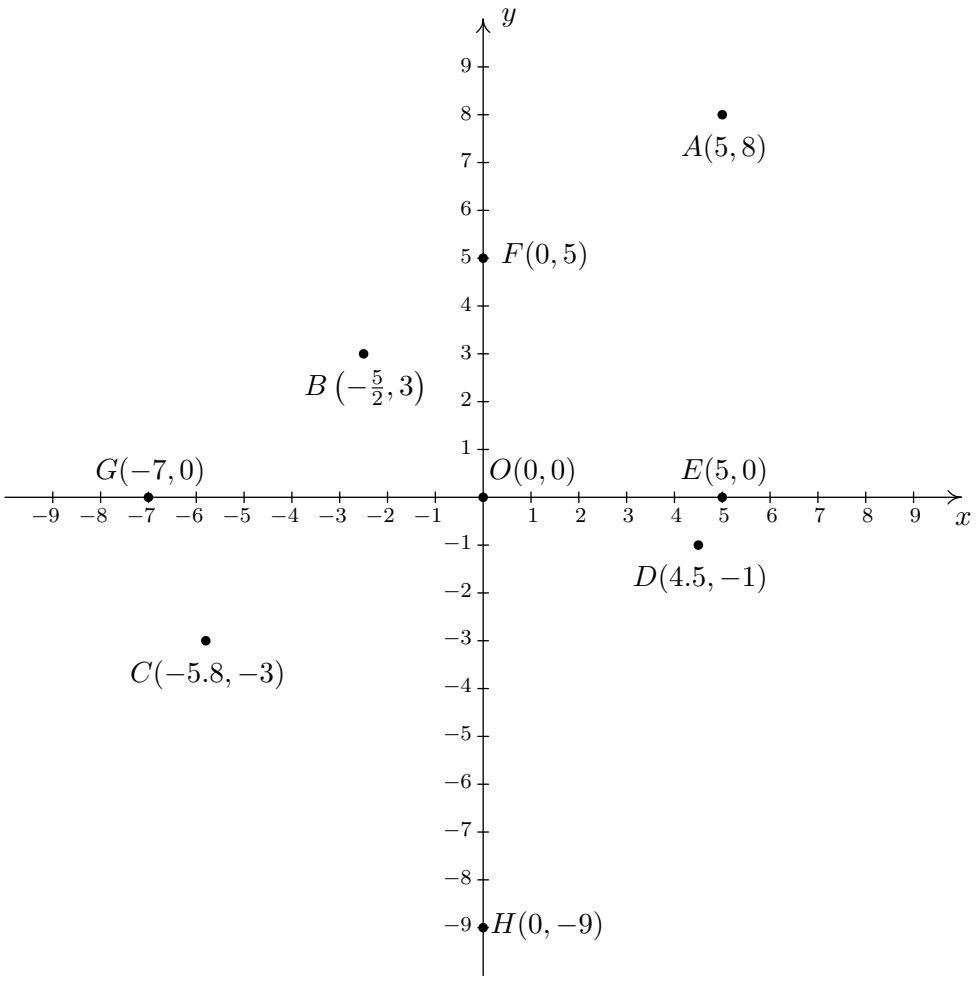



EXAMPLE 1.1.1. Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.[4]

SOLUTION. To plot these points, we start at the origin and move to the right if the $x$-coordinate is positive; to the left if it is negative. Next, we move up if the $y$-coordinate is positive or down if it is negative. If the $x$-coordinate is 0, we start at the origin and move along the $y$-axis only. If the $y$-coordinate is 0 we move along the $x$-axis only.

[3] Again, the names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the $t$-axis. The first number in the ordered pair would then be the $t$-coordinate.

[4] The letter $O$ is almost always reserved for the origin.



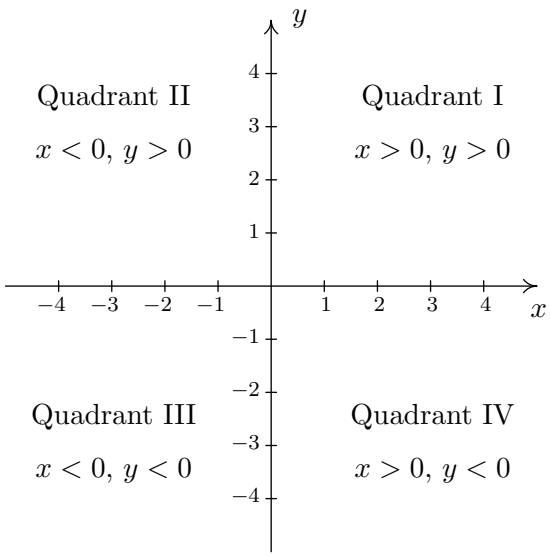

□

When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs $(x, y)$ as $x$ and $y$ take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

**Important Facts about the Cartesian Coordinate Plane**

- $(a, b)$ and $(c, d)$ represent the same point in the plane if and only if $a = c$ and $b = d$.
- $(x, y)$ lies on the $x$-axis if and only if $y = 0$.
- $(x, y)$ lies on the $y$-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



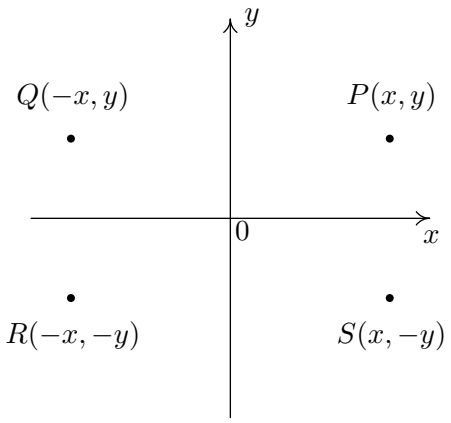

For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III, and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to the point as lying on the positive or negative $x$-axis (if $y = 0$) or on the positive or negative $y$-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive $y$-axis whereas $(-117, 0)$ lies on the negative $x$-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of mathematics is **symmetry**.There are many types of symmetry in mathematics, but three of them can be discussed easily using Cartesian Coordinates.

DEFINITION 1.1. Two points $(a, b)$ and $(c, d)$ in the plane are said to be

- **symmetric about the $x$-axis** if $a = c$ and $b = -d$
- **symmetric about the $y$-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,




In the above figure, $P$ and $S$ are symmetric about the $x$-axis, as are $Q$ and $R$; $P$ and $Q$ are symmetric about the $y$-axis, as are $R$ and $S$; and $P$ and $R$ are symmetric about the origin, as are $Q$ and $S$.

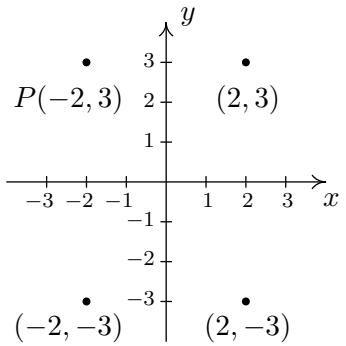
Example 1.1.2. Let $P$ be the point $(-2, 3)$. Find the points which are symmetric to $P$ about the:

1. $x$-axis
2. $y$-axis
3. origin

Check your answer by graphing.

Solution. The figure after Definition 1.1 gives us a good way to think about finding symmetric points in terms of taking the opposites of the $x$- and/or $y$-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the $x$-axis, we replace the $y$-coordinate with its opposite to get $(-2, -3)$.
2. To find the point symmetric about the $y$-axis, we replace the $x$-coordinate with its opposite to get $(2, 3)$.
3. To find the point symmetric about the origin, we replace the $x$- **and** $y$-coordinates with their opposites to get $(2, -3)$.



□

One way to visualize the processes in the previous example is with the concept of **reflections**. If we start with our point $(-2, 3)$ and pretend the $x$-axis is a mirror, then the reflection of $(-2, 3)$ across the $x$-axis would lie at $(-2, -3)$. If we pretend the $y$-axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the $x$-axis and then the $y$-axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

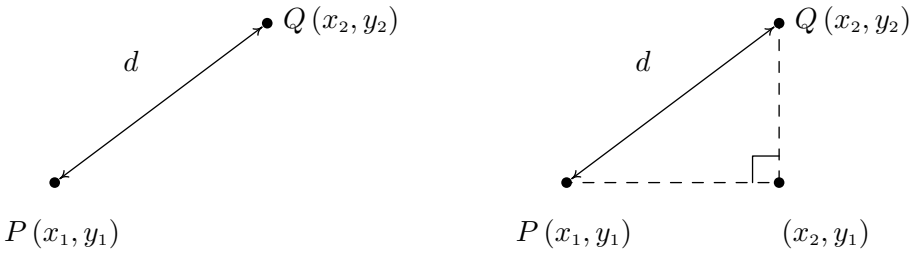
**Reflections**

To reflect a point $(x, y)$ about the:

- $x$-axis, replace $y$ with $-y$.
- $y$-axis, replace $x$ with $-x$.
- origin, replace $x$ with $-x$ and $y$ with $-y$.

### 1.1.1 Distance in the Plane

Another important concept in geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$, in the plane. By the **distance** $d$ between $P$ and $Q$, we mean the length of the line segment joining $P$ with $Q$. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.




With a little more imagination, we can envision a right triangle whose hypotenuse has length $d$ as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_2 - x_1|$ and $|y_2 - y_1|$ so the Pythagorean Theorem gives us

$$|x_2 - x_1|^2 + |y_2 - y_1|^2 = d^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Equation 1.1. **The Distance Formula:** The distance $d$ between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is not always the case that the points $P$ and $Q$ lend themselves to constructing such a triangle. If the points $P$ and $Q$ are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader to verify Equation 1.1 for these cases.

Example 1.1.3. Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

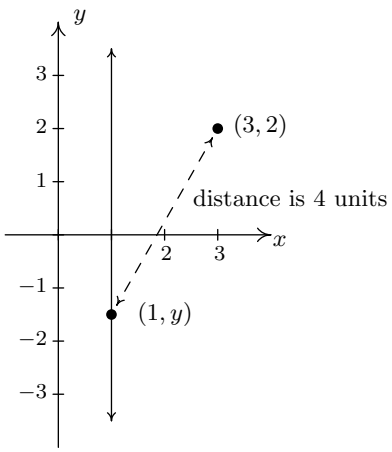
Solution.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So, the distance is $3\sqrt{5}$. □

Example 1.1.4. Find all of the points with $x$-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution. We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$. Visually,

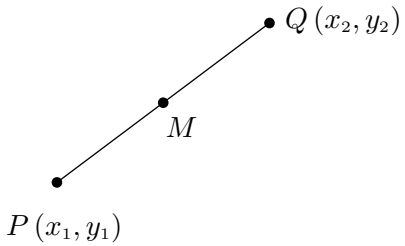



We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 1.1, yields

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}$$

We obtain two answers: $(1, 2 + 2\sqrt{3})$ and $(1, 2 - 2\sqrt{3})$. The reader is encouraged to think about why there are two answers. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$, the **midpoint**, $M$, of $P$ and $Q$ is defined to be the point on the line segment connecting $P$ and $Q$ whose distance from $P$ is equal to its distance from $Q$.




If we think of reaching $M$ by going 'halfway over' and 'halfway up' we get the following formula.

Equation 1.2. **The Midpoint Formula:** The midpoint $M$ of the line segment connecting $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$ is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

If we let $d$ denote the distance between $P$ and $Q$, we leave it as an exercise to show that the distance between $P$ and $M$ is $d/2$ which is the same as the distance between $M$ and $Q$. This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.5. Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

Solution.

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2}\right) \\ &= \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

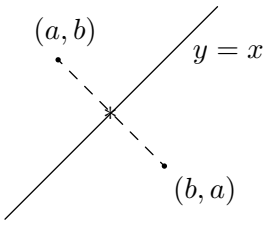
The midpoint is $\left(-\frac{1}{2}, 0\right)$. □

An interesting application[5] of the midpoint formula follows.

[5] This is a key concept in the development of inverse functions. See Section 6.2

EXAMPLE 1.1.6. Prove that the points $(a, b)$ and $(b, a)$ are symmetric about the line $y = x$.

SOLUTION. By 'symmetric about the line $y = x$', we mean that if a mirror were placed along the line $y = x$, the points $(a, b)$ and $(b, a)$ would be mirror images of one another. (You should compare and contrast this with the other types of symmetry presented back in Definition 1.1.) Schematically,



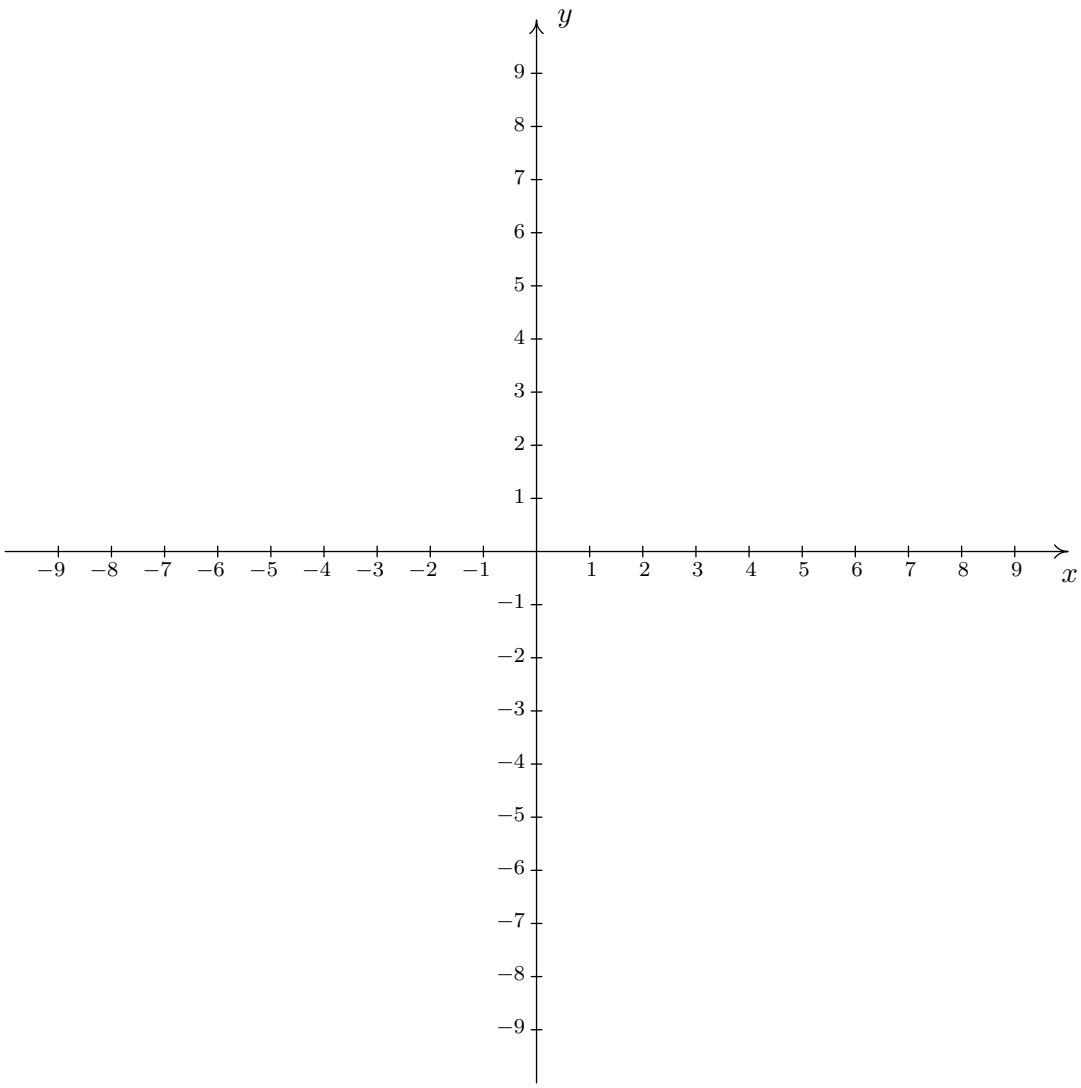

From the figure, we see that this problem amounts to showing that the midpoint of the line segment connecting $(a, b)$ and $(b, a)$ lies on the line $y = x$. Applying Equation 1.2 yields

$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the $x$ and $y$ coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required. □

### 1.1.2 Exercises

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.




2. For each point given in Exercise 1 above

   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the $x$-axis.
   - Find the point symmetric to the given point about the $y$-axis.
   - Find the point symmetric to the given point about the origin.

3. For each of the following pairs of points, find the distance $d$ between them and find the midpoint $M$ of the line segment connecting them.

   (a) $(1, 2)$, $(-3, 5)$
   (b) $(3, -10)$, $(-1, 2)$
   (c) $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$
   (d) $\left(-\frac{2}{3}, \frac{3}{2}\right)$, $\left(\frac{7}{3}, 2\right)$
   (e) $\left(\frac{24}{5}, \frac{6}{5}\right)$, $\left(-\frac{11}{5}, -\frac{19}{5}\right)$.
   (f) $\left(\sqrt{2}, \sqrt{3}\right)$, $\left(-\sqrt{8}, -\sqrt{12}\right)$
   (g) $\left(2\sqrt{45}, \sqrt{12}\right)$, $\left(\sqrt{20}, \sqrt{27}\right)$.
   (h) $(0, 0)$, $(x, y)$

4. Find all of the points of the form $(x, -1)$ which are 4 units from the point $(3, 2)$.

5. Find all of the points on the $y$-axis which are 5 units from the point $(-5, 3)$.

6. Find all of the points on the $x$-axis which are 2 units from the point $(-1, 1)$.

7. Find all of the points of the form $(x, -x)$ which are 1 unit from the origin.

8. Let's assume for a moment that we are standing at the origin and the positive $y$-axis points due North while the positive $x$-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

9. Verify the Distance Formula 1.1 for the cases when:

   (a) The points are arranged vertically. (Hint: Use $P(a, y_1)$ and $Q(a, y_2)$.)
   (b) The points are arranged horizontally. (Hint: Use $P(x_1, b)$ and $Q(x_2, b)$.)
   (c) The points are actually the same point. (You shouldn't need a hint for this one.)

10. Verify the Midpoint Formula by showing the distance between $P(x_1, y_1)$ and $M$ and the distance between $M$ and $Q(x_2, y_2)$ are both half of the distance between $P$ and $Q$.

11. Show that the points $A$, $B$ and $C$ below are the vertices of a right triangle.

    (a) $A(-3, 2)$, $B(-6, 4)$, and $C(1, 8)$
    (b) $A(-3, 1)$, $B(4, 0)$ and $C(0, -3)$

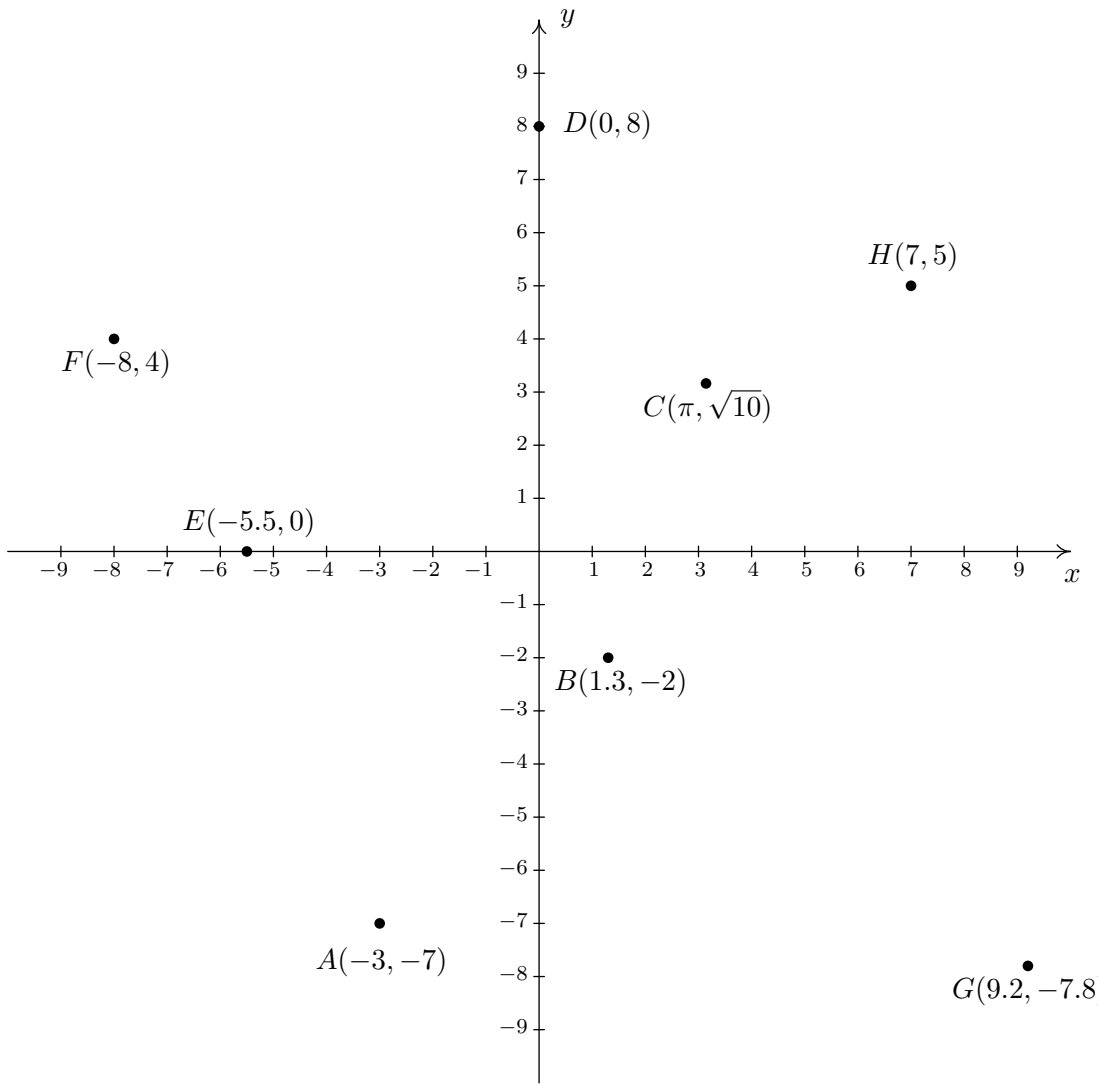
12. Find a point $D(x, y)$ such that the points $A(-3, 1)$, $B(4, 0)$, $C(0, -3)$ and $D$ are the corners of a square. Justify your answer.

13. The world is not flat.[6] Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

[6] There are those who disagree with this statement. Look them up on the Internet some time when you're bored.

### 1.1.3 Answers

1. The required points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$, and $H(7,5)$ are plotted in the Cartesian Coordinate Plane below.




2. (a) The point $A(-3,-7)$ is
   - in Quadrant III
   - symmetric about $x$-axis with $(-3,7)$
   - symmetric about $y$-axis with $(3,-7)$
   - symmetric about origin with $(3,7)$

   (b) The point $B(1.3,-2)$ is
   - in Quadrant IV
   - symmetric about $x$-axis with $(1.3,2)$
   - symmetric about $y$-axis with $(-1.3,-2)$
   - symmetric about origin with $(-1.3,2)$

   (c) The point $C(\pi,\sqrt{10})$ is
   - in Quadrant I
   - symmetric about $x$-axis with $(\pi,-\sqrt{10})$
   - symmetric about $y$-axis with $(-\pi,\sqrt{10})$

- symmetric about origin with $(-\pi, -\sqrt{10})$

(d) The point $D(0, 8)$ is
- on the positive $y$-axis
- symmetric about $x$-axis with $(0, -8)$
- symmetric about $y$-axis with $(0, 8)$
- symmetric about origin with $(0, -8)$

(e) The point $E(-5.5, 0)$ is
- on the negative $x$-axis
- symmetric about $x$-axis with $(-5.5, 0)$
- symmetric about $y$-axis with $(5.5, 0)$
- symmetric about origin with $(5.5, 0)$

(f) The point $F(-8, 4)$ is
- in Quadrant II
- symmetric about $x$-axis with $(-8, -4)$
- symmetric about $y$-axis with $(8, 4)$
- symmetric about origin with $(8, -4)$

(g) The point $G(9.2, -7.8)$ is
- in Quadrant IV
- symmetric about $x$-axis with $(9.2, 7.8)$
- symmetric about $y$-axis with $(-9.2, -7.8)$
- symmetric about origin with $(-9.2, 7.8)$

(h) The point $H(7, 5)$ is
- in Quadrant I
- symmetric about $x$-axis with $(7, -5)$
- symmetric about $y$-axis with $(-7, 5)$
- symmetric about origin with $(-7, -5)$

3. (a) $d = 5$, $M = \left(-1, \frac{7}{2}\right)$

(b) $d = 4\sqrt{10}$, $M = (1, -4)$

(c) $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$

(d) $d = \frac{\sqrt{37}}{2}$, $M = \left(\frac{5}{6}, \frac{7}{4}\right)$

(e) $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$.

(f) $d = 3\sqrt{5}$, $M = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$

(g) $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$.

(h) $d = \sqrt{x^2 + y^2}$, $M = \left(\frac{x}{2}, \frac{y}{2}\right)$

4. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$

5. $(0, 3)$

6. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$

7. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

8. $(-3, -4)$, 5 miles, $(4, -4)$

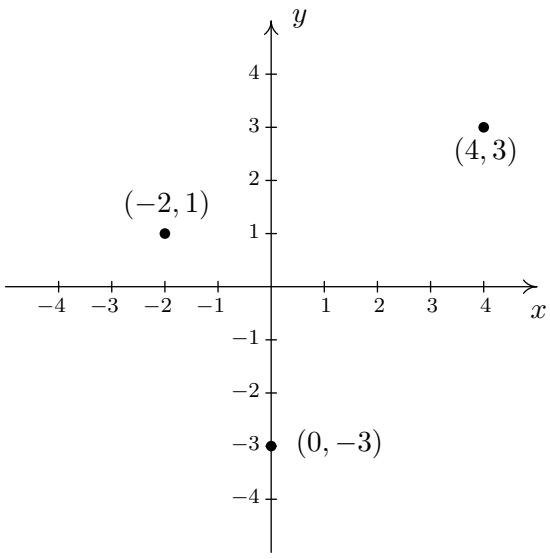
10. (a) The distance from $A$ to $B$ is $\sqrt{13}$, the distance from $A$ to $C$ is $\sqrt{52}$, and the distance from $B$ to $C$ is $\sqrt{65}$. Since
$$\left(\sqrt{13}\right)^2 + \left(\sqrt{52}\right)^2 = \left(\sqrt{65}\right)^2,$$
we are guaranteed by the converse of the Pythagorean Theorem that the triangle is right.

## 1.2 RELATIONS

We now turn our attention to sets of points in the plane.

DEFINITION 1.2. A **relation** is a set of points in the plane.

Throughout this text we will see many different ways to describe relations. In this section we will focus our attention on describing relations graphically, by means of the list (or roster) method and algebraically. Depending on the situation, one method may be easier or more convenient to use than another. Consider the set of points below




These three points constitute a relation. Let us call this relation $R$. Above, we have a **graphical** description of $R$. Although it is quite pleasing to the eye, it isn't the most portable way to describe $R$. The **list** (or **roster**) method of describing $R$ simply lists all of the points which belong to $R$. Hence, we write: $R = \{(-2, 1), (4, 3), (0, -3)\}$.[1] The roster method can be extended to describe infinitely many points, as the next example illustrates.

Example 1.2.1. Graph the following relations.

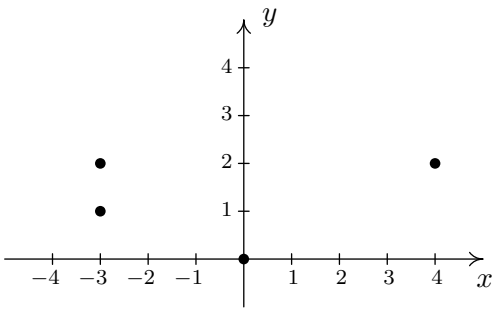
1. $A = \{(0, 0), (-3, 1), (4, 2), (-3, 2)\}$
2. $HLS_1 = \{(x, 3) : -2 \leq x \leq 4\}$
3. $HLS_2 = \{(x, 3) : -2 \leq x < 4\}$
4. $V = \{(3, y) : y \text{ is a real number}\}$

Solution.

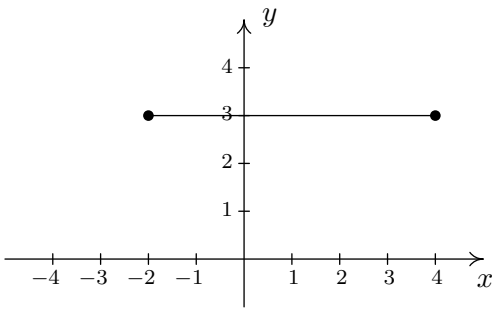
1. To graph $A$, we simply plot all of the points which belong to $A$, as shown below on the left.
2. Don't let the notation in this part fool you. The name of this relation is $HLS_1$, just like the name of the relation in part 1 was $R$. The letters and numbers are just part of its name, just

[1] We use 'set braces' {} to indicate that the points in the list all belong to the same set, in this case, $R$.

like the numbers and letters of the phrase 'King George III' were part of George's name. The next hurdle to overcome is the description of $HLS_1$ itself – a variable and some seemingly extraneous punctuation have found their way into our nice little roster notation! The way to make sense of the construction $\{(x, 3) : -2 \leq x \leq 4\}$ is to verbalize the set braces $\{\}$ as 'the set of' and the colon : as 'such that'. In words, $\{(x, 3) : -2 \leq x \leq 4\}$ is: 'the set of points $(x, 3)$ such that $-2 \leq x \leq 4$.' The purpose of the variable $x$ in this case is to describe infinitely many points. All of these points have the same $y$-coordinate, 3, but the $x$-coordinate is allowed to vary between $-2$ and 4, inclusive. Some of the points which belong to $HLS_1$ include some friendly points like: $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$, $(2, 3)$, $(3, 3)$, and $(4, 3)$. However, $HLS_1$ also contains the points $(0.829, 3)$, $\left(-\frac{5}{6}, 3\right)$, $(\sqrt{\pi}, 3)$, and so on. It is impossible to list all of these points, which is why the variable $x$ is used. Plotting several friendly representative points should convince you that $HLS_1$ describes the horizontal line segment from the point $(-2, 3)$ up to and including the point $(4, 3)$.

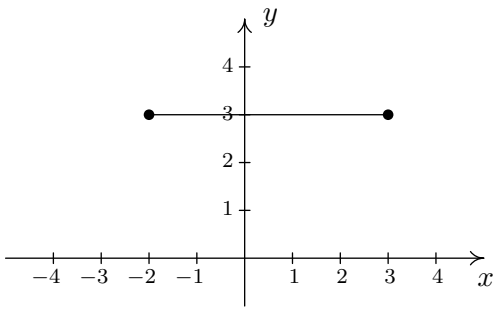



The graph of $A$

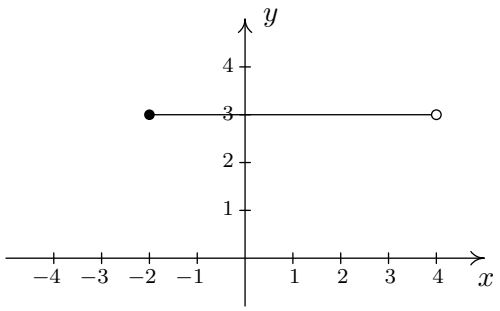



The graph of $HLS_1$

3. $HLS_2$ is hauntingly similar to $HLS_1$. In fact, the only difference between the two is that instead of '$-2 \leq x \leq 4$' we have '$-2 \leq x < 4$'. This means that we still get a horizontal line segment which includes $(-2, 3)$ and extends to $(4, 3)$, **but does not include** $(4, 3)$ because of the strict inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2, 3)$ end at $(3, 3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1, 3)$, $(3.5, 3)$, $(3.9, 3)$, $(3.99, 3)$, and so forth. There is no real number that comes 'immediately before' 4, and so to describe the set of points we want, we draw the horizontal line segment starting at $(-2, 3)$ and draw an 'open circle' at $(4, 3)$ as depicted below on the right.

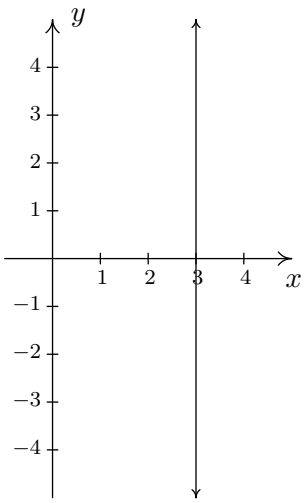



This is NOT the correct graph of $HLS_2$

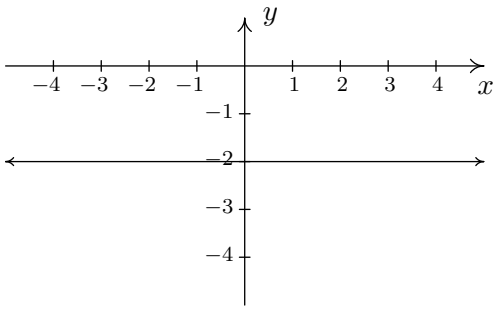



The graph of $HLS_2$

4. Our last example, $V$, describes the set of points $(3, y)$ such that $y$ is a real number. All of these points have an $x$-coordinate of 3, but the $y$-coordinate is free to be whatever it wants to be, without restriction. Plotting a few 'friendly' points of $V$ should convince you that all the points of $V$ lie on a vertical line which crosses the $x$-axis at $x = 3$. Since there is no restriction on the $y$-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of $V$ is below on the left.




The graph of $V$




The graph of $y = -2$

□

The relation $V$ in the previous example leads us to our final way to describe relations: **algebraically**. We can simply describe the points in $V$ as those points which satisfy the equation $x = 3$. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for $x$ and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose $x$-coordinate is 3. Similarly, the equation $y = -2$ in this context corresponds to all points in the plane whose $y$-coordinate is $-2$. Since there are no restrictions on the $x$-coordinate listed, we would graph the relation $y = -2$ as the horizontal line above on the right. In general, we have the following.

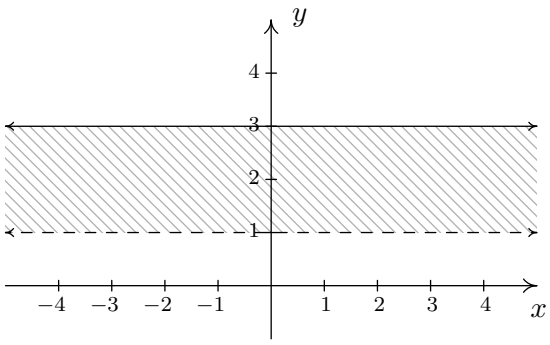
**Equations of Vertical and Horizontal Lines**

- The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

In the next section, and in many more after that, we shall explore the graphs of equations in great detail.[2] For now, we shall use our final example to illustrate how relations can be used to describe entire regions in the plane.

EXAMPLE 1.2.2. Graph the relation: $R = \{(x, y) : 1 < y \leq 3\}$

SOLUTION. The relation $R$ consists of those points whose $y$-coordinate only is restricted between 1 and 3 excluding 1, but including 3. The $x$-coordinate is free to be whatever we like. After plotting some[3] friendly elements of $R$, it should become clear that $R$ consists of the region between the horizontal lines $y = 1$ and $y = 3$. Since $R$ requires that the $y$-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to $R$. Graphically,




The graph of $R$

□

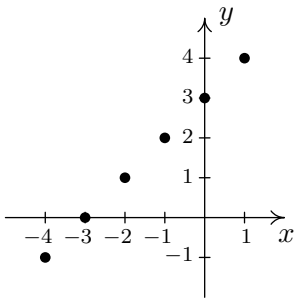
### 1.2.1 EXERCISES

1. Graph the following relations.
   (a) $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$
   (b) $\{(-2, 2),\ (-2, -1),\ (3, 5),\ (3, -4)\}$
   (c) $\left\{\left(n, 4 - n^2\right) : n = 0, \pm 1, \pm 2\right\}$
   (d) $\left\{\left(\frac{6}{k}, k\right) : k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$
2. Graph the following relations.

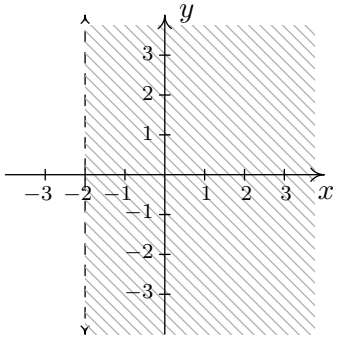
[2] In fact, much of our time in College Algebra will be spent examining the graphs of equations.
[3] The word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern.

(a) $\{(x, -2) : x > -4\}$
(b) $\{(2, y) : y \leq 5\}$
(c) $\{(-2, y) : -3 < y < 4\}$
(d) $\{(x, y) : x \leq 3\}$
(e) $\{(x, y) : y < 4\}$
(f) $\{(x, y) : x \leq 3,\ y < 2\}$
(g) $\{(x, y) : x > 0,\ y < 4\}$
(h) $\{(x, y) : -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

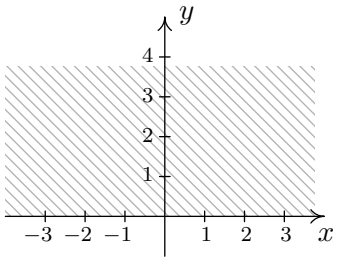
3. Describe the following relations using the roster method.



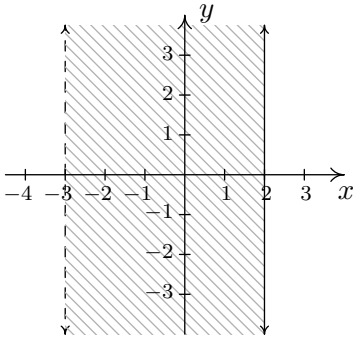

(a) The graph of relation A



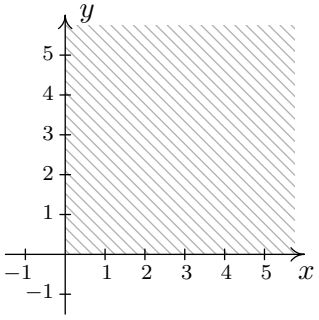

(b) The graph of relation $B$



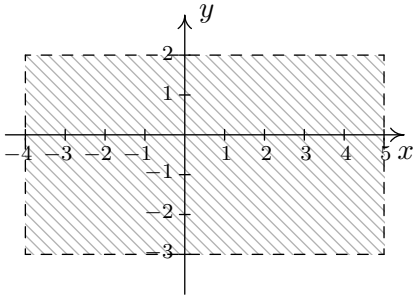

(c) The graph of relation $C$




(d) The graph of relation $D$




(e) The graph of relation $E$




(f) The graph of relation $F$

4. Graph the following lines.

(a) $x = -2$ (b) $y = 3$

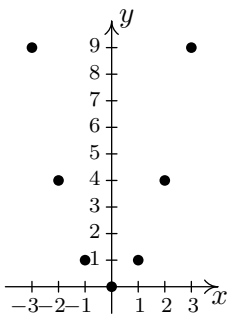
5. What is another name for the line $x = 0$? For $y = 0$?

6. Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the following relations. Please note that in the notation below we are using the **ellipsis**, ..., to denote that the list does not end, but rather, continues to follow the established pattern indefinitely. For the first two relations, give two examples of points which belong to the relation and two points which do not belong to the relation.

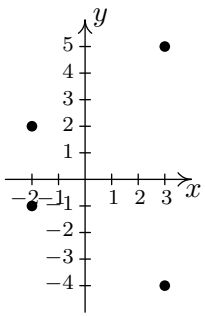
   (a) $\{(x, y) : x \text{ is an odd integer, and } y \text{ is an even integer.}\}$
   (b) $\{(x, 1) : x \text{ is an irrational number }\}$
   (c) $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$
   (d) $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

### 1.2.2 ANSWERS

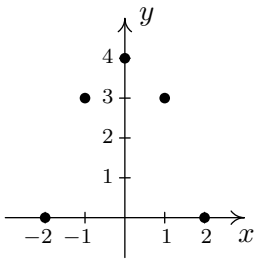
1. (a)



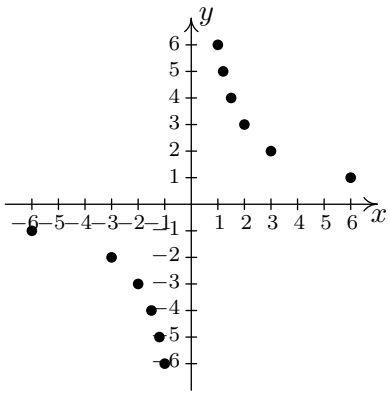

(b)




(c)

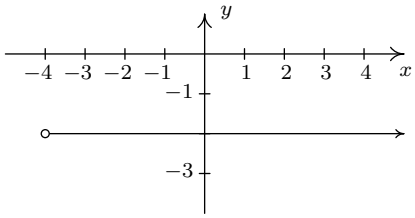



(d)

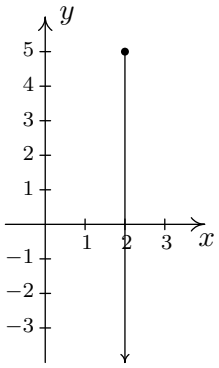



2.

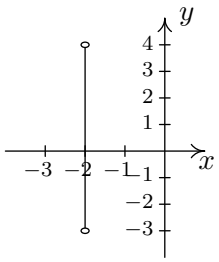
(a)



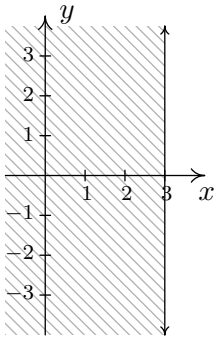

(b)



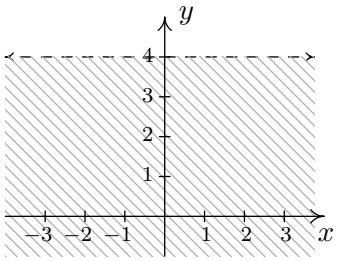

(c)



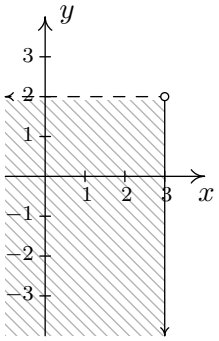

(d)



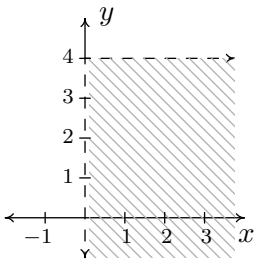

(e)



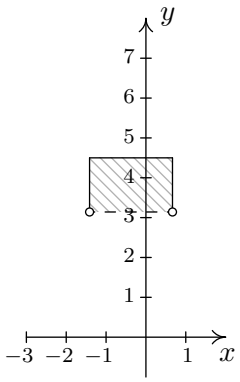

(f)




(g)




(h)




3. (a) $A = \{(-4,-1),(-3,0),(-2,1),(-1,2),(0,3),(1,4)\}$

(b) $B = \{(x, y) : x > -2\}$

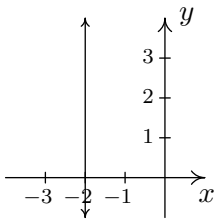
(c) $C = \{(x, y) : y \geq 0\}$

(d) $D = \{(x, y) : -3 < x \leq 2\}$

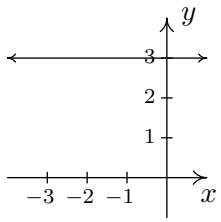
(e) $E = \{(x, y) : x \geq 0, y \geq 0\}$

(f) $F = \{(x, y) : -4 < x < 5,\ -3 < y < 2\}$

4. (a) The line $x = -2$




(b) The line $y = 3$




5. The line $x = 0$ is the $y$-axis and the line $y = 0$ is the $x$-axis.

## 1.3 Graphs of Equations

In the previous section, we said that an equation in $x$ and $y$ determines a relation.[1] In this section, we begin to explore this topic in greater detail. The main idea of this section is

**The Fundamental Graphing Principle**
The graph of an equation is the set of points which satisfy the equation. That is, a point $(x, y)$ is on the graph of an equation if and only if $x$ and $y$ satisfy the equation.

Example 1.3.1. Determine if $(2, -1)$ is on the graph of $x^2 + y^3 = 1$.

Solution. To check, we substitute $x = 2$ and $y = -1$ into the equation and see if the equation is satisfied

$$(2)^2 + (-1)^3 \stackrel{?}{=} 1$$

$$3 \neq 1$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

[1] An inequalities in $x$ and $y$ also determines a relation.

Example 1.3.2. Graph $x^2 + y^3 = 1$.

Solution. To efficiently generate points on the graph of this equation, we first solve for $y$

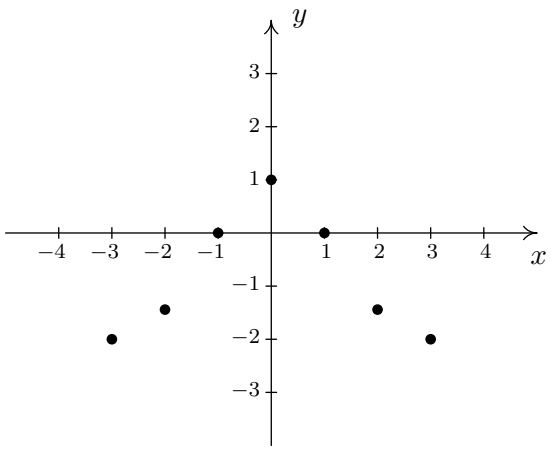
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for $x$, determine the corresponding value $y$, and plot the resulting point, $(x, y)$. For example, for $x = -3$, we substitute

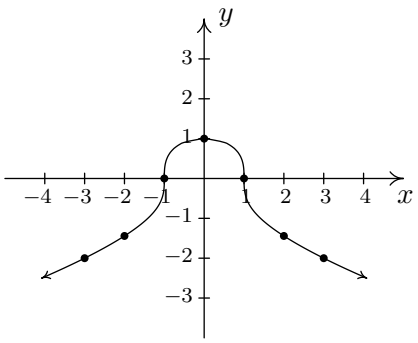
$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown below.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | $-2$ | $(-3, -2)$ |
| $-2$ | $-\sqrt[3]{3}$ | $(-2, -\sqrt[3]{3})$ |
| $-1$ | $0$ | $(-1, 0)$ |
| $0$ | $1$ | $(0, 1)$ |
| $1$ | $0$ | $(1, 0)$ |
| $2$ | $-\sqrt[3]{3}$ | $(2, -\sqrt[3]{3})$ |
| $3$ | $-2$ | $(3, -2)$ |




Remember, these points constitute only a small **sampling** of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots.' Doing so would result in a curve similar to the one pictured below.




Don't worry if you don't get all of the little bends and curves just right − Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can reach for a graphing calculator, input the formula as shown above, and graph.[2] □

Of all of the points on the graph of an equation, the places where the graph crosses the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: $x$-intercepts and $y$-intercepts. They are defined below.

DEFINITION 1.3. Suppose the graph of an equation is given.

- A point at which a graph meets the $y$-axis is called an ***y*-intercept** of the graph.

In our previous example the graph had two $x$-intercepts, $(-1, 0)$ and $(1, 0)$, and one $y$-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all! Since $x$-intercepts lie on the $x$-axis, we can find them by setting $y = 0$ in the equation. Similarly, since $y$-intercepts lie on the $y$-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

[2] Remember: At UW we don't allow calculators on exams. But using them intelligently outside of class can be a great benefit.

**Steps for finding the intercepts of the graph of an equation**

Given an equation involving $x$ and $y$:

- the $x$-intercepts always have the form $(x, 0)$; to find the $x$-intercepts of the graph, set $y = 0$ and solve for $x$.
- $y$-intercepts always have the form $(0, y)$; to find the $y$-intercepts of the graph, set $x = 0$ and solve for $y$.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the $y$-axis. To actually prove this analytically, we assume $(x, y)$ is a generic point on the graph of the equation. That is, we assume $x^2 + y^3 = 1$. As we learned in Section 1.1, the point symmetric to $(x, y)$ about the $y$-axis is $(-x, y)$. To show the graph is symmetric about the $y$-axis, we need to show that $(-x, y)$ is on the graph whenever $(x, y)$ is. In other words, we need to show $(-x, y)$ satisfies the equation $x^2 + y^3 = 1$ whenever $(x, y)$ does. Substituting gives

$$\begin{aligned} (-x)^2 + (y)^3 &\stackrel{?}{=} 1 \\ x^2 + y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

When we substituted $(-x, y)$ into the equation $x^2 + y^3 = 1$, we obtained the original equation back when we simplified. This means $(-x, y)$ satisfies the equation and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.1. The results are summarized below.

**Steps for testing if the graph of an equation possesses symmetry**

To test the graph of an equation for symmetry

- About the $y$-axis: Substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the $y$-axis.
- About the $x$-axis: Substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the $x$-axis.
- About the origin: Substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as evidenced in the next example.

EXAMPLE 1.3.3. Find the $x$- and $y$-intercepts (if any) of the graph of $(x-2)^2+y^2=1$. Test for symmetry. Plot additional points as needed to complete the graph.

SOLUTION. To look for $x$-intercepts, we set $y=0$ and solve:

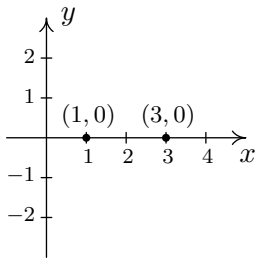
$$\begin{aligned}
(x-2)^2+y^2 &= 1 \\
(x-2)^2+0^2 &= 1 \\
(x-2)^2 &= 1 \\
\sqrt{(x-2)^2} &= \sqrt{1} \qquad \text{extract square roots} \\
x-2 &= \pm 1 \\
x &= 2 \pm 1 \\
x &= 3, 1
\end{aligned}$$

We get **two** answers for $x$ which correspond to **two** $x$-intercepts: $(1,0)$ and $(3,0)$. Turning our attention to $y$-intercepts, we set $x=0$ and solve:

$$\begin{aligned}
(x-2)^2+y^2 &= 1 \\
(0-2)^2+y^2 &= 1 \\
4+y^2 &= 1 \\
y^2 &= -3
\end{aligned}$$

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has **no** $y$-intercepts.

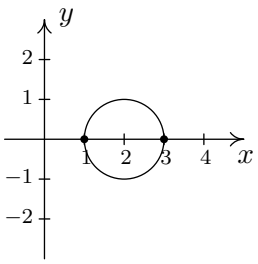
Plotting the data we have so far, we get



Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the $y$-axis or the origin. If the graph possessed either of these symmetries, then the fact that $(1, 0)$ is on the graph would mean $(-1, 0)$ would have to be on the graph. (Why?) Since $(-1, 0)$ would be another $x$-intercept (and we've found all of these), the graph can't have $y$-axis or origin symmetry. The only symmetry left to test is symmetry about the $x$-axis. To that end, we substitute $(x, -y)$ into the equation and simplify

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (x-2)^2 + (-y)^2 &\stackrel{?}{=} 1 \\ (x-2)^2 + y^2 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we have obtained our original equation, we know the graph is symmetric about the $x$-axis. This means we can cut our 'plug and plot' time in half: whatever happens below the $x$-axis is reflected above the $x$-axis, and vice-versa. Proceeding as we did in the previous example, we obtain




□

A couple of remarks are in order. First, it is entirely possible to choose a value for $x$ which does not correspond to a point on the graph. For example, in the previous example, if we solve for $y$ as is our custom, we get:

$$y = \pm\sqrt{1 - (x-2)^2}.$$

Upon substituting $x = 0$ into the equation, we would obtain

$$y = \pm\sqrt{1 - (0-2)^2} = \pm\sqrt{1-4} = \pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an $x$-coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book developing techniques which allow us to graph entire families of equations quickly.[3] Second, it is instructive to show what would have happened had we tested the equation in the last

[3] Without the use of a calculator, if you can believe it!

example for symmetry about the $y$-axis. Substituting $(-x, y)$ into the equation yields

$$\begin{aligned}(x-2)^2+y^2 &= 1\\ (-x-2)^2+y^2 &\stackrel{?}{=} 1\\ ((-1)(x+2))^2+y^2 &\stackrel{?}{=} 1\\ (x+2)^2+y^2 &\stackrel{?}{=} 1.\end{aligned}$$

This last equation does not **appear** to be equivalent to our original equation. However, to **prove** it is not symmetric about the $y$-axis, we need to find a point $(x, y)$ on the graph whose reflection $(-x, y)$ is not. Our $x$-intercept $(1, 0)$ fits this bill nicely, since if we substitute $(-1, 0)$ into the equation we get

$$\begin{aligned}(x-2)^2+y^2 &\stackrel{?}{=} 1\\ (-1-2)^2+0^2 &\neq 1\\ 9 &\neq 1.\end{aligned}$$

This proves that $(-1, 0)$ is not on the graph.

### 1.3.1 EXERCISES

1. For each equation given below

   - Find the $x$- and $y$-intercept(s) of the graph, if any exist.
   - Following the procedure in Example 1.3.2, create a table of sample points on the graph of the equation.
   - Plot the sample points and create a rough sketch of the graph of the equation.
   - Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 1.3.3

   (a) $y = x^2 + 1$
   (b) $y = x^2 - 2x - 8$
   (c) $y = x^3 - x$
   (d) $y = \frac{x^3}{4} - 3x$
   (e) $y = \sqrt{x-2}$
   (f) $y = 2\sqrt{x+4} - 2$
   (g) $3x - y = 7$
   (h) $3x - 2y = 10$
   (i) $(x+2)^2 + y^2 = 16$
   (j) $x^2 - y^2 = 1$
   (k) $4y^2 - 9x^2 = 36$
   (l) $x^3 y = -4$

2. The procedures which we have outlined in the Examples of this section and used in the exercises given above all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing these equations. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

   (a) $x^3 + y^3 - 3xy = 0$ Folium of Descartes
   (b) $x^4 = x^2 + y^2$ Kampyle of Eudoxus
   (c) $y^2 = x^3 + 3x^2$ Tschirnhausen cubic
   (d) $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

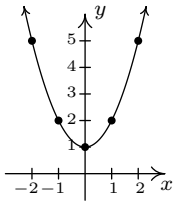
### 1.3.2 Answers

1. (a) $y = x^2 + 1$

   The graph has no $x$-intercepts

   $y$-intercept: $(0, 1)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2$ | 5 | $(-2, 5)$ |
| $-1$ | 2 | $(-1, 2)$ |
| 0 | 1 | $(0, 1)$ |
| 1 | 2 | $(1, 2)$ |
| 2 | 5 | $(2, 5)$ |




   The graph is not symmetric about the $x$-axis (e.g. $(2, 5)$ is on the graph but $(2, -5)$ is not)

   The graph is symmetric about the $y$-axis

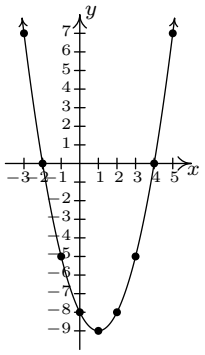
   The graph is not symmetric about the origin (e.g. $(2, 5)$ is on the graph but $(-2, -5)$ is not)

   (b) $y = x^2 - 2x - 8$

   $x$-intercepts: $(4, 0)$, $(-2, 0)$

   $y$-intercept: $(0, -8)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | 7 | $(-3, 7)$ |
| $-2$ | 0 | $(-2, 0)$ |
| $-1$ | $-5$ | $(-1, -5)$ |
| 0 | $-8$ | $(0, -8)$ |
| 1 | $-9$ | $(1, -9)$ |
| 2 | $-8$ | $(2, -8)$ |
| 3 | $-5$ | $(3, -5)$ |
| 4 | 0 | $(4, 0)$ |
| 5 | 7 | $(5, 7)$ |



The graph is not symmetric about the $x$-axis (e.g. $(-3, 7)$ is on the graph but $(-3, -7)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(-3, 7)$ is on the graph but $(3, 7)$ is not)

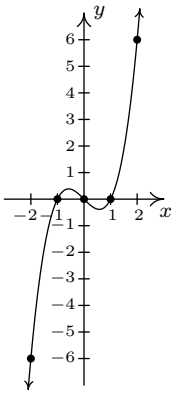
The graph is not symmetric about the origin (e.g. $(-3, 7)$ is on the graph but $(3, -7)$ is not)

(c) $y = x^3 - x$

$x$-intercepts: $(-1, 0), (0, 0), (1, 0)$

$y$-intercept: $(0, 0)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2$ | $-6$ | $(-2, -6)$ |
| $-1$ | 0 | $(-1, 0)$ |
| 0 | 0 | $(0, 0)$ |
| 1 | 0 | $(1, 0)$ |
| 2 | 6 | $(2, 6)$ |



The graph is not symmetric about the $x$-axis. (e.g. $(2, 6)$ is on the graph but $(2, -6)$ is not)

The graph is not symmetric about the $y$-axis. (e.g. $(2, 6)$ is on the graph but $(-2, 6)$ is not)

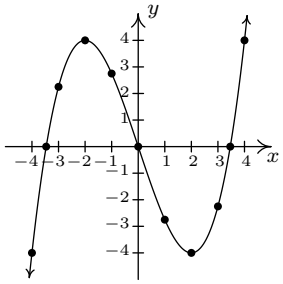
The graph is symmetric about the origin.

(d) $y = \frac{x^3}{4} - 3x$

$x$-intercepts: $\left(\pm 2\sqrt{3}, 0\right)$

$y$-intercept: $(0, 0)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-4$ | $-4$ | $(-4, -4)$ |
| $-3$ | $\frac{9}{4}$ | $\left(-3, \frac{9}{4}\right)$ |
| $-2$ | $4$ | $(-2, 4)$ |
| $-1$ | $\frac{11}{4}$ | $\left(-1, \frac{11}{4}\right)$ |
| $0$ | $0$ | $(0, 0)$ |
| $1$ | $-\frac{11}{4}$ | $\left(1, -\frac{11}{4}\right)$ |
| $2$ | $-4$ | $(2, -4)$ |
| $3$ | $-\frac{9}{4}$ | $\left(3, -\frac{9}{4}\right)$ |
| $4$ | $4$ | $(4, 4)$ |




The graph is not symmetric about the $x$-axis (e.g. $(-4, -4)$ is on the graph but $(-4, 4)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(-4, -4)$ is on the graph but $(4, -4)$ is not)

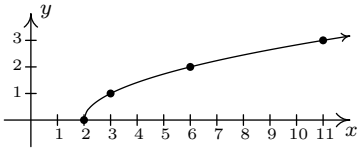
The graph is symmetric about the origin

(e) $y = \sqrt{x - 2}$

$x$-intercept: $(2, 0)$

The graph has no $y$-intercepts

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $2$ | $0$ | $(2, 0)$ |
| $3$ | $1$ | $(3, 1)$ |
| $6$ | $2$ | $(6, 2)$ |
| $11$ | $3$ | $(11, 3)$ |




The graph is not symmetric about the $x$-axis (e.g. $(3, 1)$ is on the graph but $(3, -1)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(3, 1)$ is on the graph but $(-3, 1)$ is not)

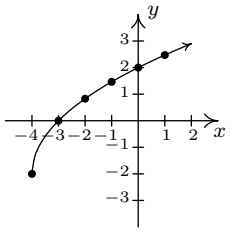
The graph is not symmetric about the origin (e.g. $(3, 1)$ is on the graph but $(-3, -1)$ is not)

(f) $y = 2\sqrt{x+4} - 2$

$x$-intercept: $(-3, 0)$

$y$-intercept: $(0, 2)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-4$ | $-2$ | $(-4, -2)$ |
| $-3$ | $0$ | $(-3, 0)$ |
| $-2$ | $2\sqrt{2} - 2$ | $\left(-2, \sqrt{2} - 2\right)$ |
| $-1$ | $2\sqrt{3} - 2$ | $\left(-2, \sqrt{3} - 2\right)$ |
| $0$ | $2$ | $(0, 2)$ |
| $1$ | $2\sqrt{5} - 2$ | $\left(-2, \sqrt{5} - 2\right)$ |




The graph is not symmetric about the $x$-axis (e.g. $(-4, -2)$ is on the graph but $(-4, 2)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(-4, -2)$ is on the graph but $(4, -2)$ is not)

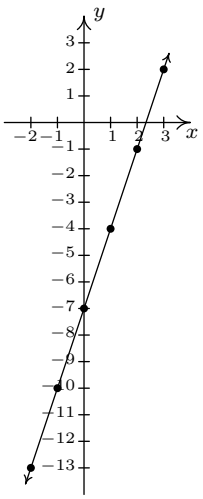
The graph is not symmetric about the origin (e.g. $(-4, -2)$ is on the graph but $(4, 2)$ is not)

(g) $3x - y = 7$
Re-write as: $y = 3x - 7$.

$x$-intercept: $(\frac{7}{3}, 0)$

$y$-intercept: $(0, -7)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2$ | $-13$ | $(-2, -13)$ |
| $-1$ | $-10$ | $(-1, -10)$ |
| $0$ | $-7$ | $(0, -7)$ |
| $1$ | $-4$ | $(1, -4)$ |
| $2$ | $-1$ | $(2, -1)$ |
| $3$ | $2$ | $(3, 2)$ |




The graph is not symmetric about the $x$-axis (e.g. $(3, 2)$ is on the graph but $(3, -2)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(3, 2)$ is on the graph but $(-3, 2)$ is not)

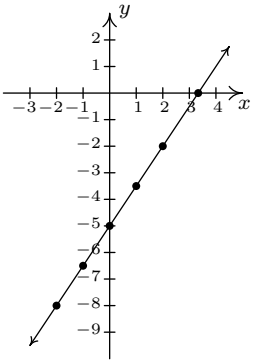
The graph is not symmetric about the origin (e.g. $(3, 2)$ is on the graph but $(-3, -2)$ is not)

(h) $3x - 2y = 10$
Re-write as: $y = \frac{3x-10}{2}$.

$x$-intercepts: $\left(\frac{10}{3}, 0\right)$

$y$-intercept: $(0, -5)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2$ | $-8$ | $(-2, -8)$ |
| $-1$ | $-\frac{13}{2}$ | $\left(-1, -\frac{13}{2}\right)$ |
| $0$ | $-5$ | $(0, -5)$ |
| $1$ | $-\frac{7}{2}$ | $\left(1, -\frac{7}{2}\right)$ |
| $2$ | $-2$ | $(2, -2)$ |



The graph is not symmetric about the $x$-axis (e.g. $(2, -2)$ is on the graph but $(2, 2)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(2, -2)$ is on the graph but $(-2, -2)$ is not)

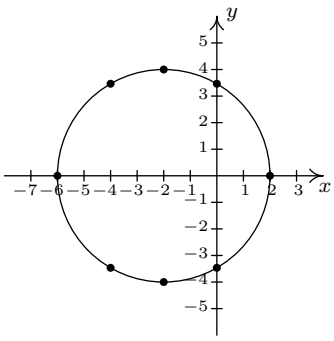
The graph is not symmetric about the origin (e.g. $(2, -2)$ is on the graph but $(-2, 2)$ is not)

(i) $(x + 2)^2 + y^2 = 16$
Re-write as $y = \pm\sqrt{16 - (x + 2)^2}$.

$x$-intercepts: $(-6, 0)$, $(2, 0)$

$y$-intercepts: $\left(0, \pm 2\sqrt{3}\right)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-6$ | $0$ | $(-6, 0)$ |
| $-4$ | $\pm 2\sqrt{3}$ | $\left(-4, \pm 2\sqrt{3}\right)$ |
| $-2$ | $\pm 4$ | $(-2, \pm 4)$ |
| $0$ | $\pm 2\sqrt{3}$ | $\left(0, \pm 2\sqrt{3}\right)$ |
| $2$ | $0$ | $(2, 0)$ |



The graph is symmetric about the $x$-axis

The graph is not symmetric about the $y$-axis (e.g. $(-6, 0)$ is on the graph but $(6, 0)$ is not)

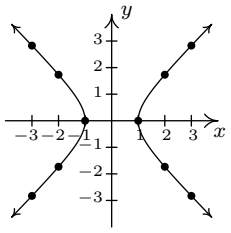
The graph is not symmetric about the origin (e.g. $(-6, 0)$ is on the graph but $(6, 0)$ is not)

(j) $x^2 - y^2 = 1$
Re-write as: $y = \pm\sqrt{x^2 - 1}$.

$x$-intercepts: $(-1, 0), (1, 0)$

The graph has no $y$-intercepts

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-3$ | $\pm\sqrt{8}$ | $(-3, \pm\sqrt{8})$ |
| $-2$ | $\pm\sqrt{3}$ | $(-2, \pm\sqrt{3})$ |
| $-1$ | $0$ | $(-1, 0)$ |
| $1$ | $0$ | $(1, 0)$ |
| $2$ | $\pm\sqrt{3}$ | $(2, \pm\sqrt{3})$ |
| $3$ | $\pm\sqrt{8}$ | $(3, \pm\sqrt{8})$ |




The graph is symmetric about the $x$-axis

The graph is symmetric about the $y$-axis

The graph is symmetric about the origin

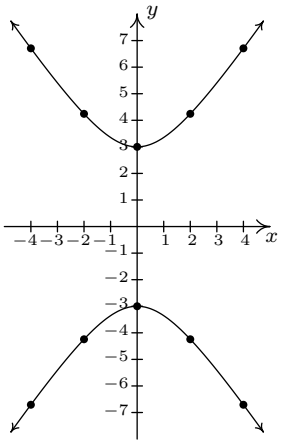
(k) $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\frac{\sqrt{9x^2+36}}{2}$.

The graph has no $x$-intercepts

$y$-intercepts: $(0, \pm 3)$

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-4$ | $\pm 3\sqrt{5}$ | $\left(-4, \pm 3\sqrt{5}\right)$ |
| $-2$ | $\pm 3\sqrt{2}$ | $\left(-2, \pm 3\sqrt{2}\right)$ |
| $0$ | $\pm 3$ | $(0, \pm 3)$ |
| $2$ | $\pm 3\sqrt{2}$ | $\left(2, \pm 3\sqrt{2}\right)$ |
| $4$ | $\pm 3\sqrt{5}$ | $\left(4, \pm 3\sqrt{5}\right)$ |




The graph is symmetric about the $x$-axis

The graph is symmetric about the $y$-axis

The graph is symmetric about the origin

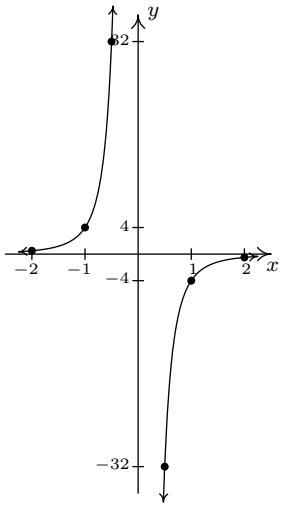
(l) $x^3y = -4$

Re-write as: $y = -\dfrac{4}{x^3}$.

The graph has no $x$-intercepts

The graph has no $y$-intercepts

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| $-2$ | $\frac{1}{2}$ | $(-2, \frac{1}{2})$ |
| $-1$ | $4$ | $(-1, 4)$ |
| $-\frac{1}{2}$ | $32$ | $(-\frac{1}{2}, 32)$ |
| $\frac{1}{2}$ | $-32$ | $(\frac{1}{2}, -32)$ |
| $1$ | $-4$ | $(1, -4)$ |
| $2$ | $-\frac{1}{2}$ | $(2, -\frac{1}{2})$ |




The graph is not symmetric about the $x$-axis (e.g. $(1, -4)$ is on the graph but $(1, 4)$ is not)

The graph is not symmetric about the $y$-axis (e.g. $(1, -4)$ is on the graph but $(-1, -4)$ is not)

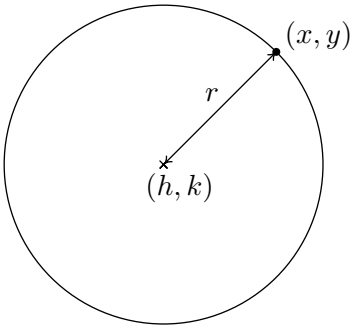
The graph is symmetric about the origin

## 1.4 Three Interesting Curves

### 1.4.1 Circles

Recall from geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 1.4. A **circle** with center $(h, k)$ and radius $r > 0$ is the set of all points $(x, y)$ in the plane whose distance to $(h, k)$ is $r$.




From the picture, we see that a point $(x, y)$ is on the circle if and only if its distance to $(h, k)$ is $r$. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

EQUATION 1.3. **The Standard Equation of a Circle:** The equation of a circle with center $(h, k)$ and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

EXAMPLE 1.4.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

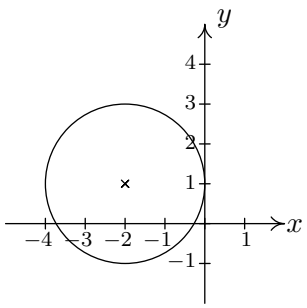
SOLUTION. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

□

EXAMPLE 1.4.2. Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

SOLUTION. From the standard form of a circle, Equation 1.3, we have that $x+2$ is $x-h$, so $h = -2$ and $y-1$ is $y-k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 1.3 by following the steps given below.

**To Put a Circle into Standard Form**

1. Group the same variables together on one side of the equation and put the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 1.4.3. Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution.

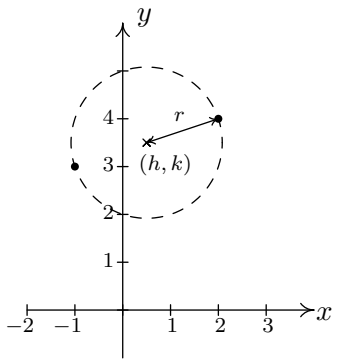
$$\begin{aligned}
3x^2-6x+3y^2+4y-4 &= 0 && \\
3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2-2x+1\right)+3\left(y^2+\frac{4}{3}y+\frac{4}{9}\right) &= 4+3(1)+3\left(\frac{4}{9}\right) && \text{complete the square in } x,\ y \\
3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}$$

From Equation 1.3, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

EXAMPLE 1.4.4. Write the standard equation of the circle which has $(-1, 3)$ and $(2, 4)$ as the endpoints of a diameter.

SOLUTION. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields




Since the given points are endpoints of a diameter, we know their midpoint $(h, k)$ is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right)\\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{10}{4}$ □

We close this section with the most important[4] circle in all of mathematics: the **Unit Circle**.

[4]While this may seem like an opinion, it is indeed a fact.

Definition 1.5. The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 1.4.5. Find the points on the unit circle with $y$-coordinate $\frac{\sqrt{3}}{2}$.

Solution. We replace $y$ with $\frac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

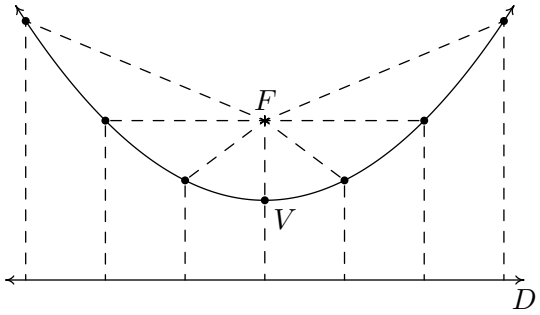
$$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1.$$

From this we get $x^2 + \frac{3}{4} = 1$ so $x^2 = \frac{1}{4}$ so $x = \pm\frac{1}{2}$. The points are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. These two points are the intersection of the horizontal line $y = \frac{\sqrt{3}}{2}$ and the unit circle $x^2 + y^2 = 1$, □

### 1.4.2 Parabolas

Definition 1.6. Let $F$ be a point in the plane and $D$ be a line not containing $F$. The set of all points equidistant from $F$ and $D$ is called the **parabola** with **focus** $F$ and **directrix** $D$.

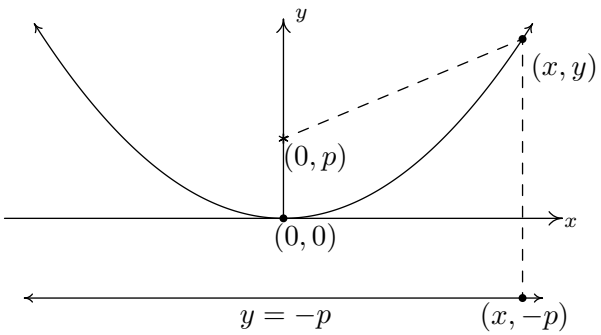
Schematically, we have the following.




Each dashed line from the point $F$ to a point on the curve has the same length as the dashed line from the point on the curve to the line $D$. The point suggestively labeled $V$ is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section

3.3. Let $p$ denote the directed[5] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes




From the definition of parabola, we know the distance from $(0, p)$ to $(x, y)$ is the same as the distance from $(x, -p)$ to $(x, y)$. Using the Distance Formula, Equation 1.1, we get

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} && \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} && \\
x^2+(y-p)^2 &= (y+p)^2 && \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 && \text{expand quantities} \\
x^2 &= 4py && \text{gather like terms}
\end{aligned}$$

Solving for $y$ yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 3.4 with $a = \frac{1}{4p}$ and vertex $(0, 0)$.

We know from previous experience that if the coefficient of $x^2$ is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that $p$ is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

What if we choose to place the vertex at an arbitrary point $(h, k)$? We can either use transformations (vertical and horizontal shifts from Section 2.5) or re-derive the equation from Definition 1.6 to arrive at the following.

[5] We'll talk more about what 'directed' means later.

Equation 1.4. **The Standard Equation of a Vertical[a] Parabola:** The equation of a (vertical) parabola with vertex $(h, k)$ and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

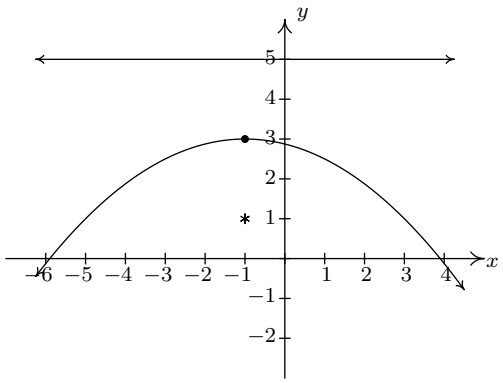
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a] That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, $x$, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

Example 1.4.6. Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

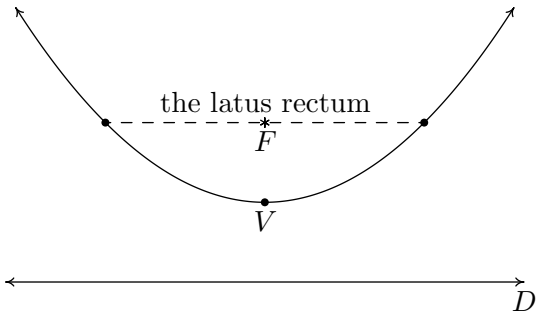
Solution. We recognize this as the form given in Equation 1.4. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-3$ so $k=3$. Hence, the vertex is $(-1, 3)$. We also see that $4p = -8$ so $p = -2$. Since $p < 0$, the focus will be below the vertex and the parabola will open downwards. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. If we start at $(-1, 3)$ and move down 2 units, we arrive at the focus $(-1, 1)$. The directrix, then, is 2 units above the vertex and if we move 2 units up from $(-1, 3)$, we'd be on the horizontal line $y = 5$.




□

Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[6] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.

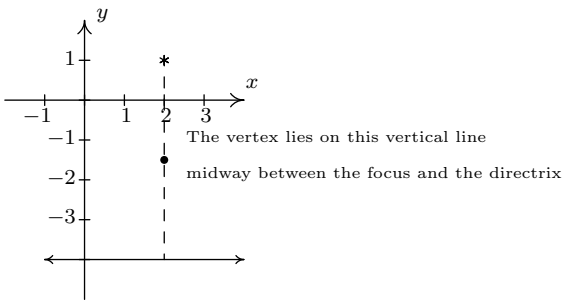
[6] No, I'm not making this up.




It turns out[7] that the length of the latus rectum is $|4p|$, which, in light of Equation 1.4, is easy to find. In our last example, for instance, when graphing $(x+1)^2 = -8(y-3)$, we can use the fact that the length of the latus rectum is $|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

EXAMPLE 1.4.7. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

SOLUTION. Sketching the data yields,



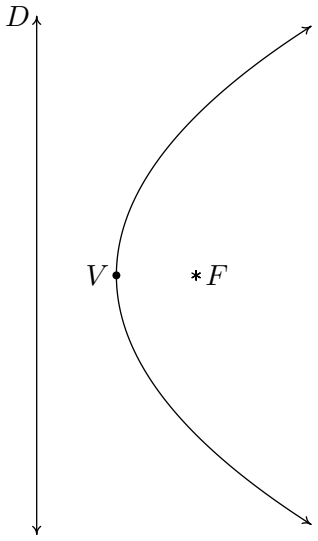

From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an $x$-coordinate of 2. To find the $y$-coordinate, we note that the distance from the focus to the directrix is $1-(-4) = 5$, which means the vertex lies $5/2$ units (halfway) below the focus. Starting at $(2, 1)$ and moving down $5/2$ units leaves us at $(2, -3/2)$, which is our vertex. Since the parabola opens upwards, we know $p$ is positive. Thus $p = 5/2$. Plugging all of this data into Equation 1.4 give us

$$\begin{aligned} (x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y-\left(-\frac{3}{2}\right)\right) \\ (x-2)^2 &= 10\left(y+\frac{3}{2}\right) \end{aligned}$$

□

[7] Consider this an exercise to show what follows.

If we interchange the roles of $x$ and $y$, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[8] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex. A typical 'horizontal' parabola is sketched below.




Equation 1.5. **The Standard Equation of a Horizontal Parabola:** The equation of a (horizontal) parabola with vertex $(h, k)$ and focal length $|p|$ is

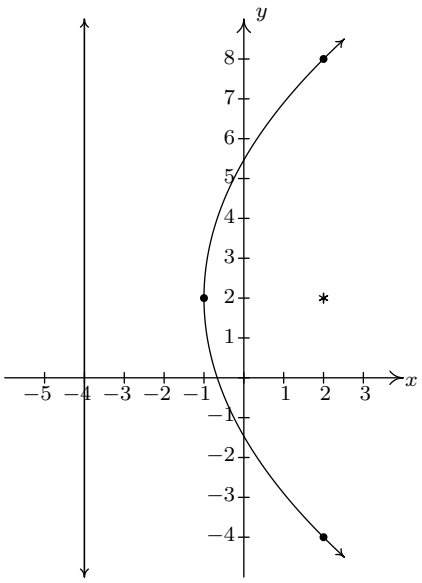
$$(y-k)^2 = 4p(x-h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Example 1.4.8. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 1.5. Here, $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 2$ so $k = 2$. Hence, the vertex is $(-1, 2)$. We also see that $4p = 12$ so $p = 3$. Since $p > 0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p| = 3$, which means the focus is 3 units to the right. If we start at $(-1, 2)$ and move right 3 units, we arrive at the focus $(2, 2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1, 2)$, we'd be on the vertical line $x = -4$. Since the length of the latus rectum is $|4p| = 12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.

[8] plural of 'directrix'




As with circles, not all parabolas will come to us in the forms in Equations 1.4 or 1.5. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

**To Put a Parabola into Standard Form**

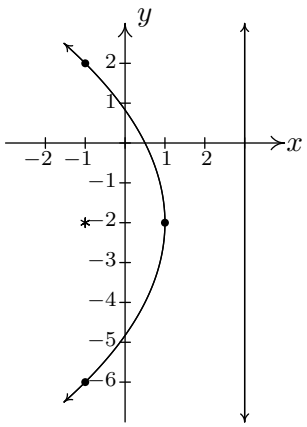
1. Group the variable which is squared on one side of the equation and put the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

EXAMPLE 1.4.9. Consider the equation $y^2 + 4y + 8x = 4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

SOLUTION. We need to get a perfect square (in this case, using $y$) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the $x$) on the other.

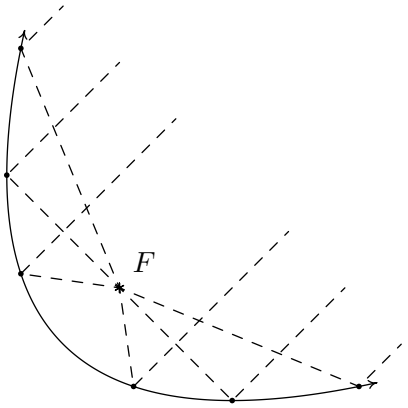
$$\begin{aligned} y^2 + 4y + 8x &= 4 \\ y^2 + 4y &= -8x + 4 \\ y^2 + 4y + 4 &= -8x + 4 + 4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x + 8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

Now that the equation is in the form given in Equation 1.5, we see that $x - h$ is $x - 1$ so $h = 1$, and $y - k$ is $y + 2$ so $k = -2$. Hence, the vertex is $(1, -2)$. We also see that $4p = -8$ so that $p = -2$. Since $p < 0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units to the left of 1, so if we start at $(1, -2)$ and move left 2 units, we arrive at the focus $(-1, -2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1, -2)$, we'd be on the vertical line $x = 3$. Since the length of the latus rectum is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



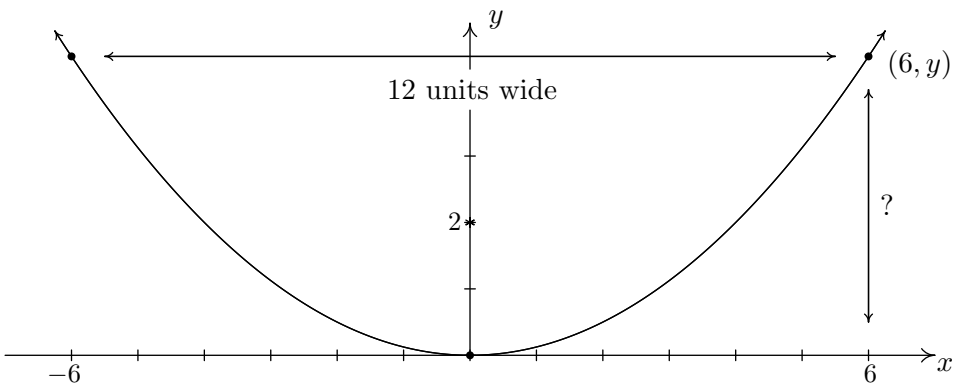

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.

Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.




EXAMPLE 1.4.10. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

SOLUTION. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0, 0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



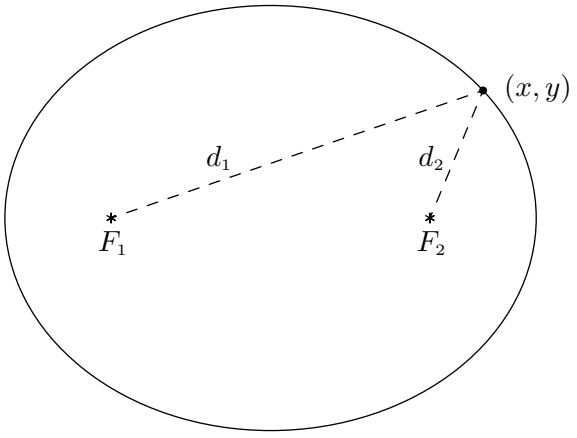

Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the $y$ value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = 36/8 = 9/2 = 4.5$. Hence, the dish will be 9/2 or 4.5 feet deep. □

### 1.4.3 Ellipses

In the definition of a circle, Definition 1.4, we fixed a point called the **center** and considered all of the points which were a fixed distance $r$ from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance $d$ to use in our definition.

Definition 1.7. Given two distinct points $F_1$ and $F_2$ in the plane and a fixed distance $d$, an **ellipse** is the set of all points $(x, y)$ in the plane such that the sum of the distance from $F_1$ to $(x, y)$ and the distance from $F_2$ to $(x, y)$ is $d$. The points $F_1$ and $F_2$ are called the **foci**[a] of the ellipse.

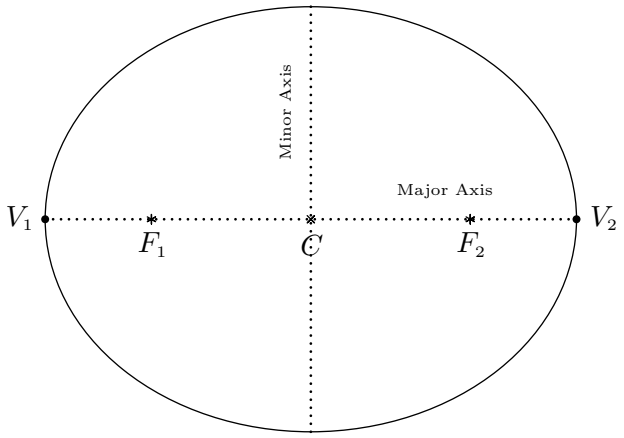
[a] the plural of 'focus'




$d_1 + d_2 = d$ for all $(x, y)$ on the ellipse

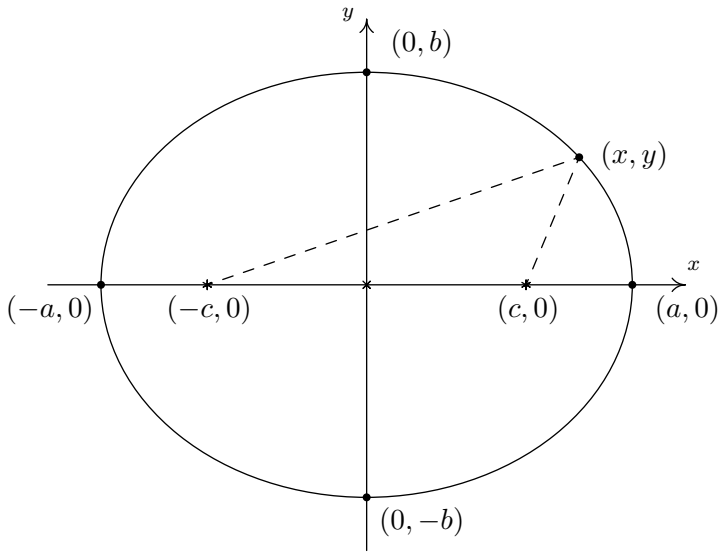
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,




An ellipse with center $C$; foci $F_1$, $F_2$; and vertices $V_1$, $V_2$

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0, 0)$, its major axis along the $x$-axis, and has foci $(c, 0)$ and $(-c, 0)$ and vertices $(-a, 0)$ and $(a, 0)$. We will label the $y$-intercepts of the ellipse as $(0, b)$ and $(0, -b)$ (We assume $a$, $b$, and $c$ are all positive numbers.) Schematically,




Note that since $(a, 0)$ is on the ellipse, it must satisfy the conditions of Definition 1.7. That is, the distance from $(-c, 0)$ to $(a, 0)$ plus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance $d$. Since all of these points lie on the $x$-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c) + (a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance $d$ mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0, b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\ \sqrt{(0-(-c))^2 + (b-0)^2} + \sqrt{(0-c)^2 + (b-0)^2} &= 2a \\ \sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\ 2\sqrt{b^2+c^2} &= 2a \\ \sqrt{b^2+c^2} &= a \end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point $(x, y)$ on the ellipse. Applying Definition 1.7, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to do some rearranging, squaring, and more rearranging.[9]

$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2-cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}$$

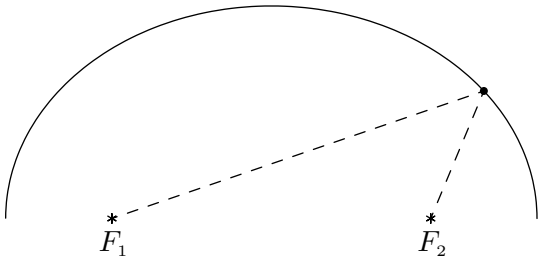
We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below as sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head

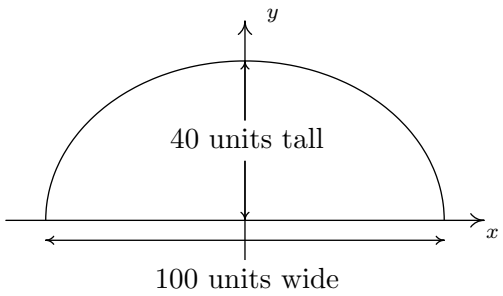
[9] In other words, tons and tons of Intermediate Algebra. Stay sharp, this is not for the faint of heart.

towards the other focus. Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them.




Example 1.4.11. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields




It's most convenient to imagine this ellipse centered at $(0, 0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation

$$\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1.$$

We're looking for the foci, and we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, so that the foci are 30 units from the center. That means they are $50 - 30 = 20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery. □

### 1.4.4 EXERCISES

1. Find the standard equation of the circle given the center and radius and sketch its graph.

   (a) Center $(-1, -5)$, radius 10
   (b) Center $(4, -2)$, radius 3
   (c) Center $\left(-3, \frac{7}{13}\right)$, radius $\frac{1}{2}$
   (d) Center $(\pi, e^2)$, radius $\sqrt[3]{91}$
   (e) Center $\left(-e, \sqrt{2}\right)$, radius $\pi$

2. Complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

   (a) $x^2 - 4x + y^2 + 10y = -25$
   (b) $-2x^2 - 36x - 2y^2 - 112 = 0$
   (c) $x^2 + y^2 + 8x - 10y - 1 = 0$
   (d) $x^2 + y^2 + 5x - y - 1 = 0$
   (e) $4x^2 + 4y^2 - 24y + 36 = 0$
   (f) $x^2 + x + y^2 - \frac{6}{5}y = 1$

3. Find the standard equation of the circle which satisfies the following criteria:

   (a) center $(3, 5)$, passes through $(-1, -2)$
   (b) center $(3, 6)$, passes through $(-1, 4)$
   (c) endpoints of a diameter: $(3, 6)$ and $(-1, 4)$
   (d) endpoints of a diameter: $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$

4. Verify the following points lie on the Unit Circle: $(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$

5. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[10] Find an equation for the wheel assuming that its center lies on the $y$-axis.

6. Sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

   (a) $(y - 2)^2 = -12(x + 3)$
   (b) $(y + 4)^2 = 4x$
   (c) $(x - 3)^2 = -16y$
   (d) $\left(x + \frac{7}{3}\right)^2 = 2\left(y + \frac{5}{2}\right)$
   (e) $(x - 1)^2 = 4(y + 3)$
   (f) $(x + 2)^2 = -20(y - 5)$
   (g) $(y - 4)^2 - 18(x - 2)$
   (h) $\left(y + \frac{3}{2}\right)^2 = -7\left(x + \frac{9}{2}\right)$

7. Put the equation into standard form and identify the vertex, focus and directrix.

[10] Source: Cedar Point's webpage.

(a) $y^2 - 10y - 27x + 133 = 0$
(b) $25x^2 + 20x + 5y - 1 = 0$
(c) $x^2 + 2x - 8y + 49 = 0$
(d) $2y^2 + 4y + x - 8 = 0$
(e) $x^2 - 10x + 12y + 1 = 0$
(f) $3y^2 - 27y + 4x + \frac{211}{4} = 0$

8. Find an equation for the parabola which fits the given criteria.

   (a) Vertex $(7, 0)$, focus $(0, 0)$
   (b) Vertex $(-8, -9)$, Both $(0, 0)$, $(-16, 0)$ are points on the curve
   (c) Focus $(10, 1)$, directrix $x = 5$
   (d) The endpoints of latus rectum are $(-2, -7)$ and $(-4, -7)$

9. Graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the foci and the eccentricity.

   (a) $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
   (b) $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
   (c) $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
   (d) $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
   (e) $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
   (f) $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
   (g) $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

10. Put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the foci and the eccentricity.

    (a) $12x^2 + 3y^2 - 30y + 39 = 0$
    (b) $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
    (c) $x^2 - 2x + 2y^2 - 12y + 3 = 0$
    (d) $9x^2 + 4y^2 - 4y - 8 = 0$
    (e) $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
    (f) $6x^2 + 5y^2 - 24x + 20y + 14 = 0$

11. Find the standard form of the equation of the ellipse which has the given properties.

    (a) Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
    (b) All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$[11]
    (c) Foci $(0, \pm 4)$, Point on curve $\left(2, \frac{5\sqrt{5}}{3}\right)$
    (d) Vertex $(-10, 5)$, Focus $(-2, 5)$, Eccentricity $\frac{1}{2}$

[11] One might also say that the ellipse is "tangent to the axes" at those two points.

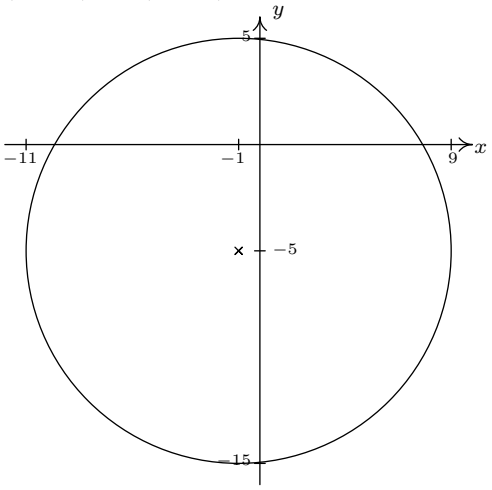
12. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farther from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

13. With the help of your classmates, research whispering galleries and other ways ellipses have been used in architecture and design.

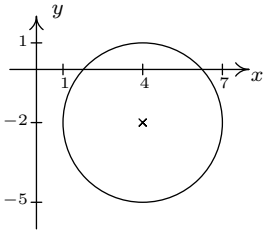
14. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

### 1.4.5 Answers

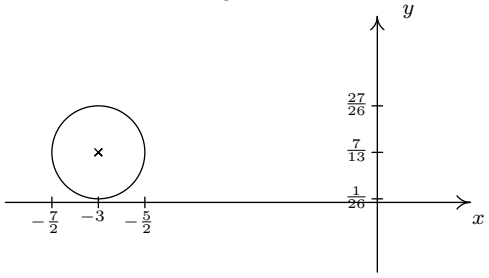
1. (a) $(x+1)^2+(y+5)^2=100$



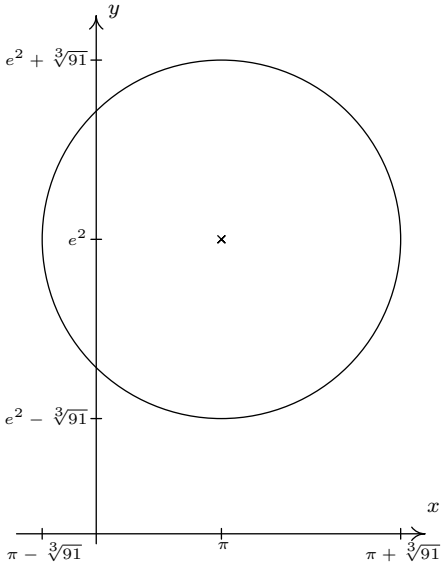

(b) $(x-4)^2+(y+2)^2=9$



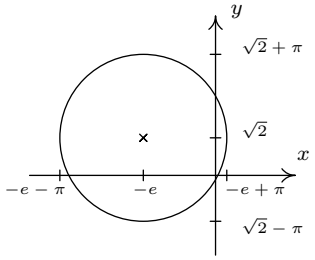

(c) $(x+3)^2+\left(y-\frac{7}{13}\right)^2=\frac{1}{4}$




(d) $(x-\pi)^2+\left(y-e^2\right)^2=91^{\frac{2}{3}}$




(e) $(x+e)^2+\left(y-\sqrt{2}\right)^2=\pi^2$




2. (a) $(x-2)^2+(y+5)^2=4$
   Center $(2,-5)$, radius $r=2$

   (b) $(x+9)^2+y^2=25$
   Center $(-9,0)$, radius $r=5$

   (c) $(x+4)^2+(y-5)^2=42$
   Center $(-4,5)$, radius $r=\sqrt{42}$

   (d) $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
   Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

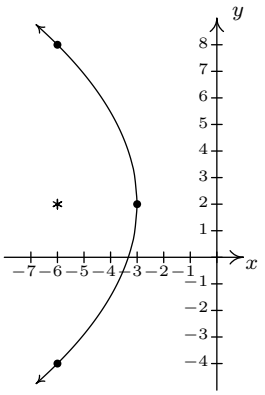
   (e) $x^2+(y-3)^2=0$
   This is not a circle.

   (f) $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
   Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

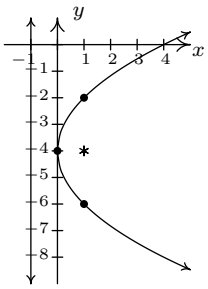
3. (a) $(x-3)^2+(y-5)^2=65$
   (b) $(x-3)^2+(y-6)^2=20$
   (c) $(x-1)^2+(y-5)^2=5$
   (d) $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$

5. $x^2+(y-72)^2=4096$

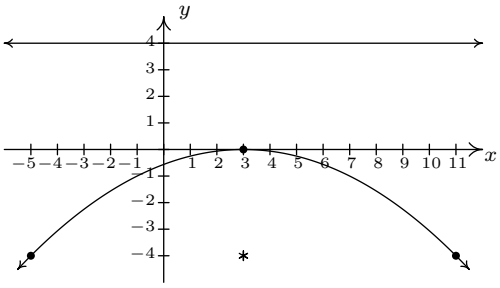
6. (a) $(y-2)^2=-12(x+3)$
   Vertex $(-3,2)$
   Focus $(-6,2)$
   Directrix $x=0$
   Endpoints of latus rectum $(-6,8)$, $(-6,-4)$



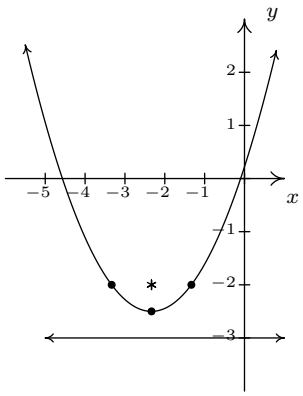
   (b) $(y+4)^2=4x$
   Vertex $(0,-4)$
   Focus $(1,-4)$
   Directrix $x=-1$
   Endpoints of latus rectum $(1,-2)$, $(1,-6)$



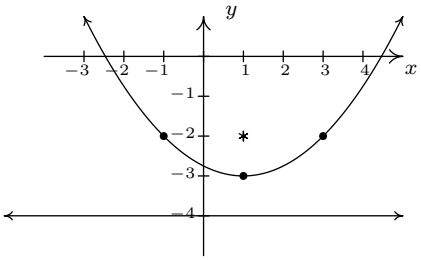
   (c) $(x-3)^2=-16y$
   Vertex $(3,0)$
   Focus $(3,-4)$
   Directrix $y=4$
   Endpoints of latus rectum $(-5,-4)$, $(11,-4)$



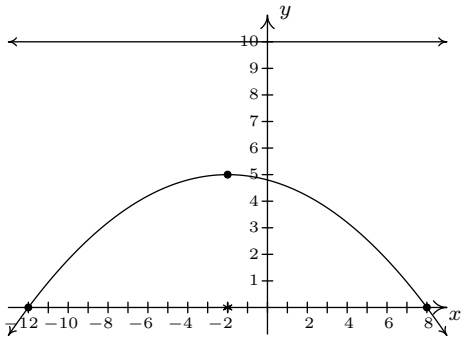
   (d) $\left(x+\frac{7}{3}\right)^2=2\left(y+\frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3},-\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3},-2\right)$
   Directrix $y=-3$
   Endpoints of latus rectum $\left(-\frac{10}{3},-2\right)$, $\left(-\frac{4}{3},-2\right)$



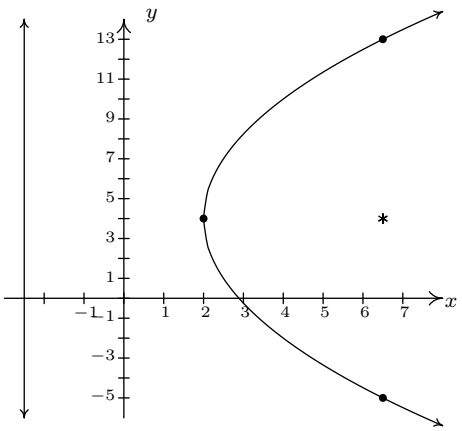

(e) $(x-1)^2 = 4(y+3)$
Vertex $(1,-3)$
Focus $(1,-2)$
Directrix $y=-4$
Endpoints of latus rectum $(3,-2)$, $(-1,-2)$



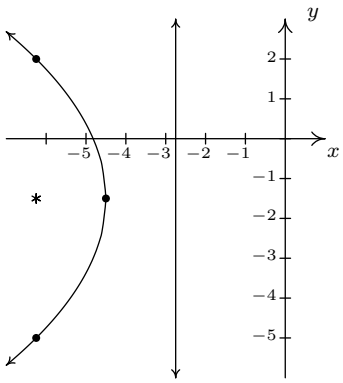

(f) $(x+2)^2 = -20(y-5)$
Vertex $(-2,5)$
Focus $(-2,0)$
Directrix $y=10$
Endpoints of latus rectum $(-12,0)$, $(8,0)$




(g) $(y-4)^2 - 18(x-2)$
Vertex $(2,4)$
Focus $\left(\frac{13}{2},4\right)$
Directrix $x=-\frac{5}{2}$
Endpoints of latus rectum $\left(-\frac{13}{2},-5\right)$, $\left(\frac{13}{2},13\right)$



(h) $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2}, -\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4}, -\frac{3}{2}\right)$
Directrix $x = -\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4}, 2\right)$, $\left(-\frac{25}{4}, -5\right)$




7. (a) $(y-5)^2 = 27(x-4)$
Vertex $(4, 5)$
Focus $\left(\frac{43}{4}, 5\right)$
Directrix $x = -\frac{11}{4}$

(b) $\left(x+\frac{2}{5}\right)^2 = -\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5}, 1\right)$
Focus $\left(-\frac{2}{5}, \frac{19}{20}\right)$
Directrix $y = \frac{21}{20}$

(c) $(x+1)^2 = 8(y-6)$
Vertex $(-1, 6)$
Focus $(-1, 8)$
Directrix $y = 4$

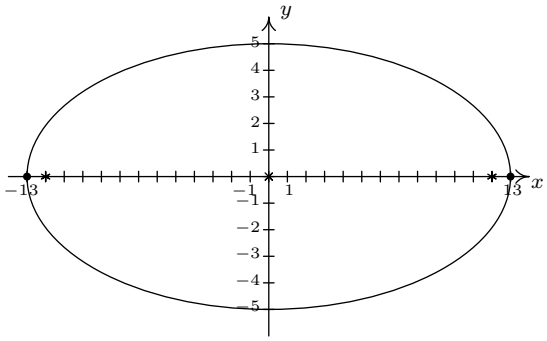
(d) $(y+1)^2 = -\frac{1}{2}(x-10)$
Vertex $(10, -1)$
Focus $\left(\frac{79}{8}, -1\right)$
Directrix $x = \frac{81}{8}$

(e) $(x-5)^2 = -12(y-2)$
Vertex $(5, 2)$
Focus $(5, -1)$
Directrix $y = 5$

(f) $\left(y-\frac{9}{2}\right)^2 = -\frac{4}{3}(x-2)$
Vertex $\left(2, \frac{9}{2}\right)$
Focus $\left(\frac{5}{3}, \frac{9}{2}\right)$
Directrix $x = \frac{7}{3}$

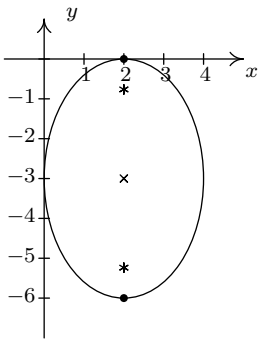
8. (a) $y^2 = -28(x-7)$
(b) $(x+8)^2 = \frac{64}{9}(y+9)$
(c) $(y-1)^2 = 10\left(x-\frac{15}{2}\right)$
(d) $(x-1)^2 = 6\left(y+\frac{17}{2}\right)$ or $(x-1)^2 = -6\left(y+\frac{11}{2}\right)$

9. (a) $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
Center $(0, 0)$
Major axis along $y = 0$
Minor axis along $x = 0$
Vertices $(13, 0)$, $(-13, 0)$
Foci $(12, 0)$, $(-12, 0)$
$e = \dfrac{12}{13}$

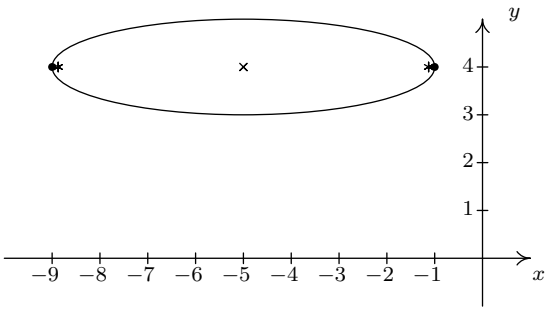



(b) $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
Center $(2, -3)$
Major axis along $x = 2$
Minor axis along $y = -3$
Vertices $(2, 0)$, $(2, -6)$

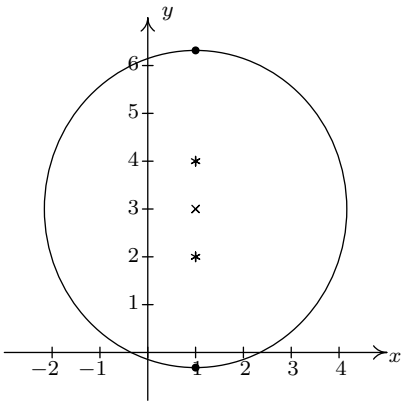
Foci $(2, -3+\sqrt{5})$, $(2, -3-\sqrt{5})$
$e = \dfrac{\sqrt{5}}{3}$



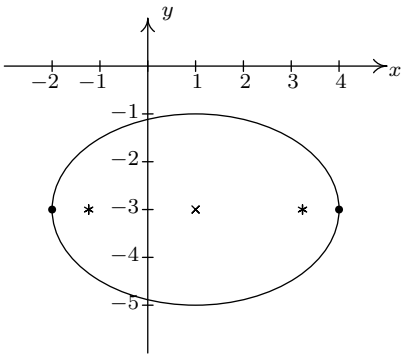

(c) $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
Center $(-5, 4)$
Major axis along $y = 4$
Minor axis along $x = -5$
Vertices $(-9, 4)$, $(-1, 4)$
Foci $(-5+\sqrt{15}, 4)$, $(-5-\sqrt{15}, 4)$
$e = \dfrac{\sqrt{15}}{4}$




(d) $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1, 3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Foci $(1, 2)$, $(1, 4)$
$e = \dfrac{\sqrt{11}}{11}$




(e) $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1, -3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4, -3)$, $(-2, -3)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \dfrac{\sqrt{5}}{3}$



(f) $\dfrac{(x+2)^2}{16}+\dfrac{(y-5)^2}{20}=1$

Center $(-2,5)$

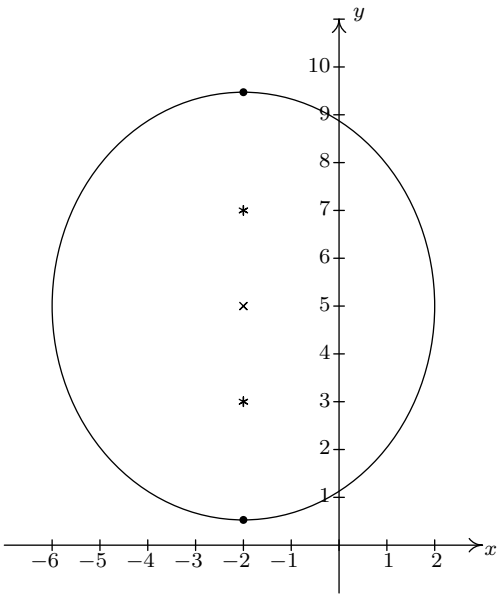
Major axis along $x=-2$

Minor axis along $y=5$

Vertices $(-2,5+2\sqrt{5})$, $(-2,5-2\sqrt{5})$

Foci $(-2,7)$, $(-2,3)$

$e=\dfrac{\sqrt{5}}{5}$




(g) $\dfrac{(x-4)^2}{8}+\dfrac{(y-2)^2}{18}=1$

Center $(4,2)$

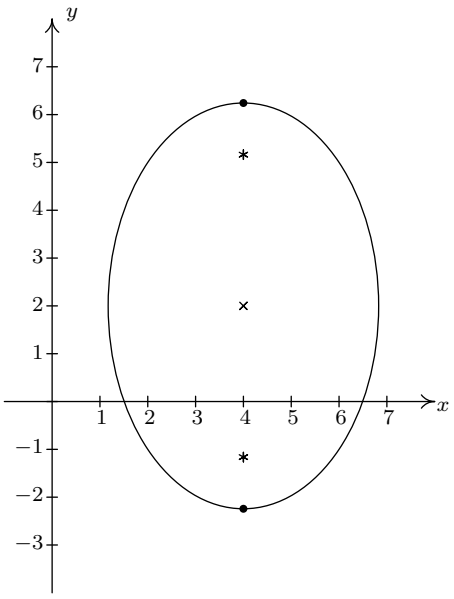
Major axis along $x=4$

Minor axis along $y=2$

Vertices $(4,2+3\sqrt{2})$, $(4,2-3\sqrt{2})$

Foci $(4,2+\sqrt{10})$, $(4,2-\sqrt{10})$

$e=\dfrac{\sqrt{5}}{3}$




10. (a) $\dfrac{x^2}{3}+\dfrac{(y-5)^2}{12}=1$

Center $(0,5)$

Major axis along $x=0$

Minor axis along $y=5$

Vertices $(0,5-2\sqrt{3}),(0,5+2\sqrt{3})$

Foci $(0,2),(0,8)$

$e=\frac{\sqrt{3}}{2}$

(b) $\dfrac{(x-3)^2}{18}+\dfrac{(y+2)^2}{5}=1$

Center $(3,-2)$

Major axis along $y=-2$

Minor axis along $x=3$

Vertices $(3-3\sqrt{2},-2),(3+3\sqrt{2},-2)$

Foci $(3-\sqrt{13},-2),(3+\sqrt{13},-2)$

$e=\frac{\sqrt{26}}{6}$

(c) $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{8}=1$
Center $(1,3)$
Major Axis along $y=3$
Minor Axis along $x=1$
Vertices $(5,3)$, $(-3,3)$
Foci $(1+2\sqrt{2},3)$, $(1-2\sqrt{2},3)$
$e=\frac{\sqrt{2}}{2}$

(d) $\dfrac{x^2}{1}+\dfrac{4\left(y-\frac{1}{2}\right)^2}{9}=1$
Center $\left(0,\frac{1}{2}\right)$
Major Axis along $x=0$ (the $y$-axis)
Minor Axis along $y=\frac{1}{2}$
Vertices $(0,2)$, $(0,-1)$
Foci $\left(0,\frac{1+\sqrt{5}}{2}\right)$, $\left(0,\frac{1-\sqrt{5}}{2}\right)$
$e=\frac{\sqrt{5}}{3}$

(e) $\dfrac{(x-3)^2}{25}+\dfrac{(y-1)^2}{9}=1$
Center $(3,1)$
Major Axis along $y=1$
Minor Axis along $x=3$
Vertices $(8,1)$, $(-2,1)$
Foci $(7,1)$, $(-1,1)$
$e=\frac{4}{5}$

(f) $\dfrac{(x-2)^2}{5}+\dfrac{(y+2)^2}{6}=1$
Center $(2,-2)$
Major Axis along $x=2$
Minor Axis along $y=-2$
Vertices $\left(2,-2+\sqrt{6}\right)$, $(2,-2-\sqrt{6})$
Foci $(2,-1)$, $(2,-3)$
$e=\frac{\sqrt{6}}{6}$

11. (a) $\dfrac{(x-3)^2}{9}+\dfrac{(y-7)^2}{25}=1$

(b) $\dfrac{(x-8)^2}{64}+\dfrac{(y+9)^2}{81}=1$

(c) $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$

(d) $\dfrac{(x-6)^2}{256}+\dfrac{(y-5)^2}{192}=1$

12. Distance from the sun to aphelion $\approx 1.0167$ AU.
Distance from the sun to perihelion $\approx 0.9833$ AU.

# Chapter 2

# Functions

## 2.1 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 2.1. A relation in which each $x$-coordinate is matched with only one $y$-coordinate is said to describe $y$ as a **function** of $x$.

Example 2.1.1. Which of the following relations describe $y$ as a function of $x$?

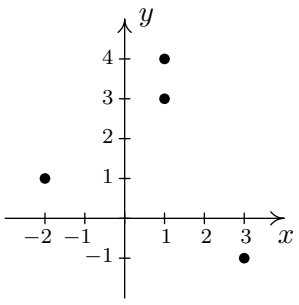
1. $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
2. $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$

Solution. A quick scan of the points in $R_1$ reveals that the $x$-coordinate 1 is matched with two **different** $y$-coordinates: namely 3 and 4. Hence in $R_1$, $y$ is **not** a function of $x$. On the other hand, every $x$-coordinate in $R_2$ occurs only once which means each $x$-coordinate has only one corresponding $y$-coordinate. So, $R_2$ **does** represent $y$ as a function of $x$. □

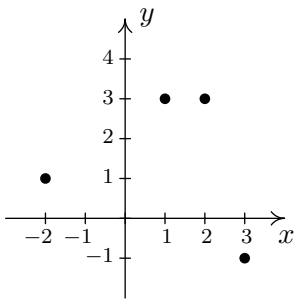
Note that in the previous example, the relation $R_2$ contained two different points with the same $\boldsymbol{y}$-coordinates, namely $(1, 3)$ and $(2, 3)$. Remember, in order to say $y$ is a function of $x$, we just need to ensure the same $x$-coordinate isn't used in more than one point.[1]

To see what the function concept means geometrically, we graph $R_1$ and $R_2$ in the plane.

[1] We will have occasion later in the text to concern ourselves with the concept of $x$ being a function of $y$. In this case, $R_1$ represents $x$ as a function of $y$; $R_2$ does not.




The graph of $R_1$



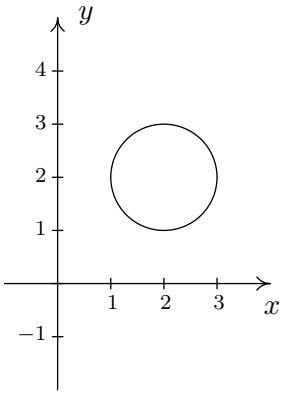

The graph of $R_2$

The fact that the $x$-coordinate 1 is matched with two different $y$-coordinates in $R_1$ presents itself graphically as the points $(1, 3)$ and $(1, 4)$ lying on the same vertical line, $x = 1$. If we turn our attention to the graph of $R_2$, we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

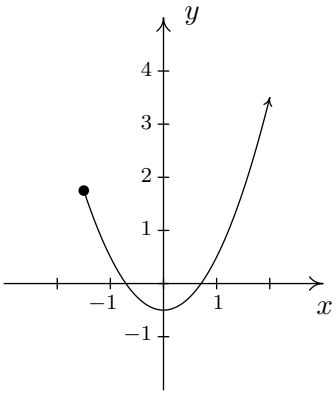
THEOREM 2.1. **The Vertical Line Test:** A set of points in the plane represents $y$ as a function of $x$ if and only if no two points lie on the same vertical line.

It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of 'function' as well as the concept of 'graph'.

EXAMPLE 2.1.2. Use the Vertical Line Test to determine which of the following relations describes $y$ as a function of $x$.




The graph of $R$

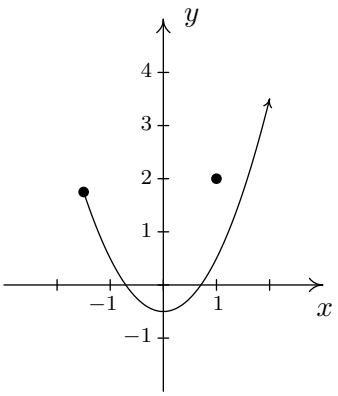



The graph of $S$

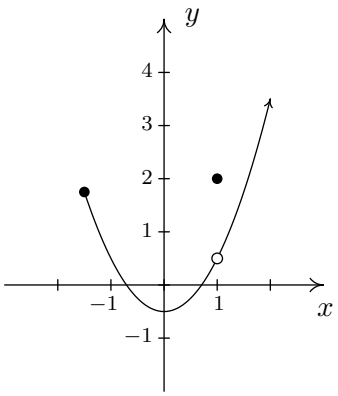
SOLUTION. Looking at the graph of $R$, we can easily imagine a vertical line crossing the graph more than once. Hence, $R$ does **not** represent $y$ as a function of $x$. However, in the graph of $S$, every vertical line crosses the graph at most once, and so $S$ **does** represent $y$ as a function of $x$. □

In the previous test, we say that the graph of the relation $R$ **fails** the Vertical Line Test, whereas the graph of $S$ **passes** the Vertical Line Test. Note that in the graph of $R$ there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is **one** vertical line that fits the bill, as the next example illustrates.

Example 2.1.3. Use the Vertical Line Test to determine which of the following relations describes $y$ as a function of $x$.

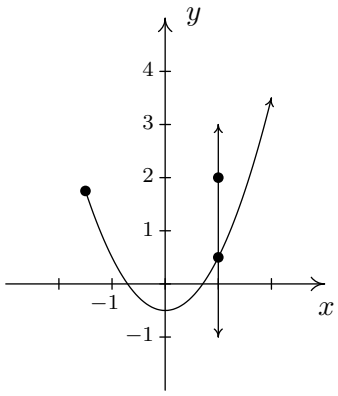



The graph of $S_1$




The graph of $S_2$

Solution. Both $S_1$ and $S_2$ are slight modifications to the relation $S$ in the previous example whose graph we determined passed the Vertical Line Test. In both $S_1$ and $S_2$, it is the addition of the point $(1, 2)$ which threatens to cause trouble. In $S_1$, there is a point on the curve with $x$-coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below.) Hence, the graph of $S_1$ fails the Vertical Line Test, so $y$ is **not** a function of $x$ here. However, in $S_2$ notice that the point with $x$-coordinate 1 on the curve has been omitted, leaving an 'open circle' there. Hence, the vertical line $x = 1$ crosses the graph of $S_2$ only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we have that the graph of $S_2$ passes the Vertical Line Test. Thus it describes $y$ as a function of $x$.




$S_1$ and the line $x = 1$

□

Suppose a relation $F$ describes $y$ as a function of $x$. The sets of $x$- and $y$-coordinates are given special names.

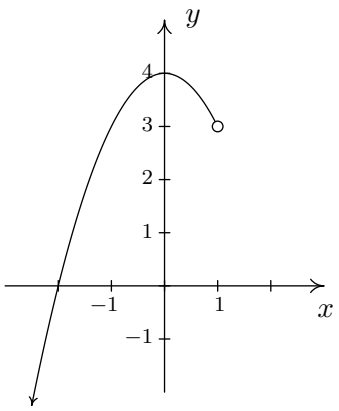
DEFINITION 2.2. Suppose $F$ is a relation which describes $y$ as a function of $x$.

- The set of the $x$-coordinates of the points in $F$ is called the **domain** of $F$.
- The set of the $y$-coordinates of the points in $F$ is called the **range** of $F$.

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

EXAMPLE 2.1.4. Find the domain and range of the following functions

1. $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$
2. $G$ is the function graphed below:



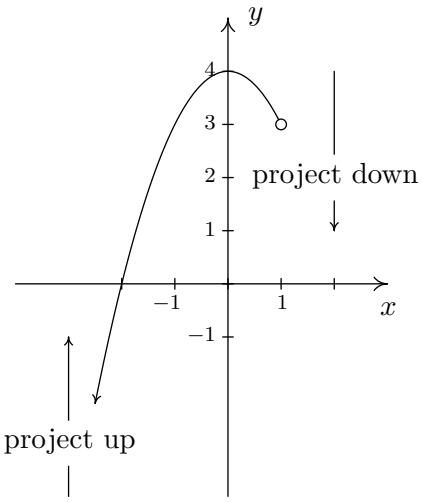

The graph of $G$

SOLUTION. The domain of $F$ is the set of the $x$-coordinates of the points in $F$: $\{-3, 0, 4, 5\}$ and the range of $F$ is the set of the $y$-coordinates: $\{1, 2\}$.[2]

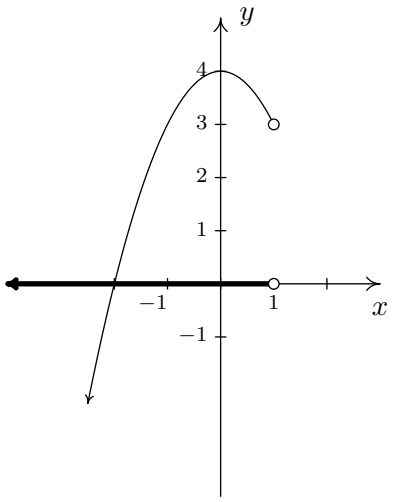
To determine the domain and range of $G$, we need to determine which $x$ and $y$ values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the $x$-axis and determining the portion of the $x$-axis that gets covered. This is called

[2]When listing numbers in a set, we list each number only once, in increasing order.

**projecting** the curve to the $x$-axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ isn't on the graph, but all points on the curve leading up to that point are on the curve.

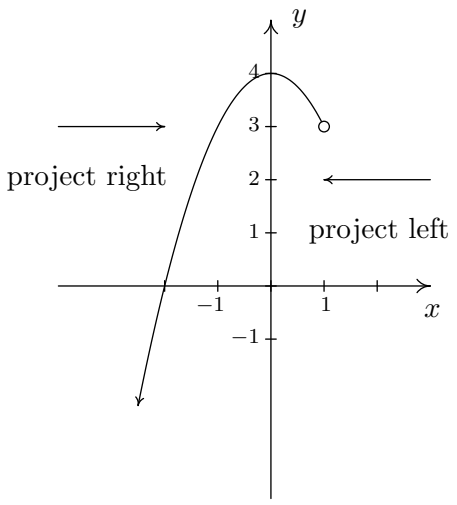



The graph of $G$

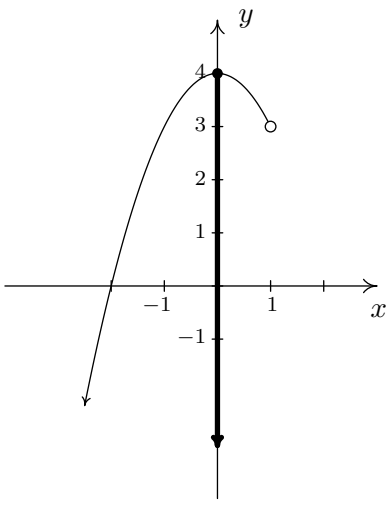



The graph of $G$

We see from the figure that if we project the graph of $G$ to the $x$-axis, we get all real numbers less than 1. Using interval notation, we write the domain of $G$ is $(-\infty, 1)$. To determine the range of $G$, we project the curve to the $y$-axis as follows:




The graph of $G$




The graph of $G$

Note that even though there is an open circle at $(1, 3)$, we still include the $y$ value of 3 in our range, since the point $(-1, 3)$ is on the graph of $G$. We see that the range of $G$ is all real numbers less than or equal to 4, or, in interval notation: $(-\infty, 4]$. □

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 1.2 may or may not describe $y$ as a function of $x$. The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 2.2.)

EXAMPLE 2.1.5. Determine which equations represent $y$ as a function of $x$:

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $x^2 y = 1 - 3y$

SOLUTION. For each of these equations, we solve for $y$ and determine whether each choice of $x$ will determine only one corresponding value of $y$.

1.
$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1 - x^3} \qquad \text{extract square roots} \\ y &= \pm\sqrt{1 - x^3} \end{aligned}$$

If we substitute $x = 0$ into our equation for $y$, we get: $y = \pm\sqrt{1 - 0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does **not** represent $y$ as a function of $x$.

2.
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

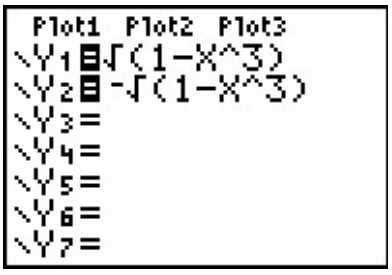
For every choice of $x$, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of $y$. Hence, this equation describes $y$ as a function of $x$.

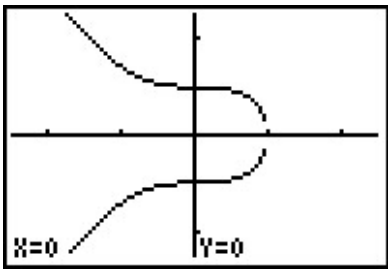
3.

$$\begin{aligned} x^2y &= 1-3y \\ x^2y+3y &= 1 \\ y\left(x^2+3\right) &= 1 \qquad \text{factor} \\ y &= \frac{1}{x^2+3} \end{aligned}$$

For each choice of $x$, there is only one value for $y$, so this equation describes $y$ as a function of $x$. □

Of course, we could always use our graphing calculator to verify our responses to the previous example. For example, if we wanted to verify that the first equation does not represent $y$ as a function of $x$, we could enter the equation for $y$ into the calculator as indicated below and graph. Note that we need to enter both solutions – the positive and the negative square root – for $y$. The resulting graph clearly fails the Vertical Line Test, so does not represent $y$ as a function of $x$.




### 2.1.1 Exercises

1. Determine which of the following relations represent $y$ as a function of $x$. Find the domain and range of those relations which are functions.

   (a) $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

   (b) $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

   (c) $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

   (d) $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

   (e) $\{(x,y) : x \text{ is an odd integer, and } y \text{ is an even integer}\}$

   (f) $\{(x,1) : x \text{ is an irrational number}\}$

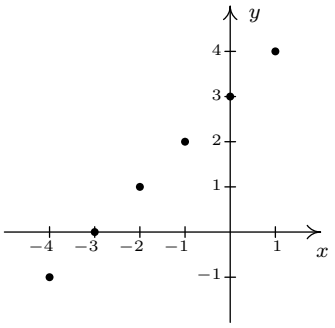
   (g) $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$

   (h) $\{\ldots,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$

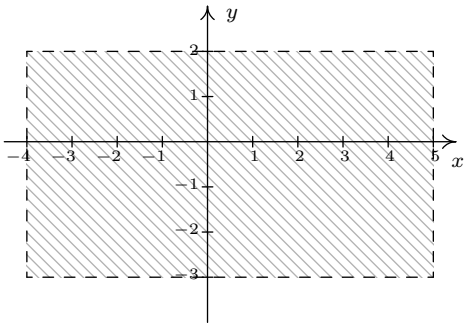
   (i) $\{(-2,y) : -3 < y < 4\}$

   (j) $\{(x,3) : -2 \le x < 4\}$

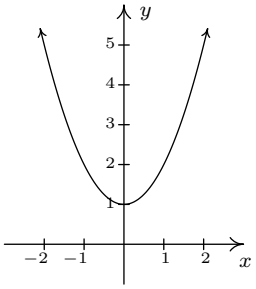
2. Determine which of the following relations represent $y$ as a function of $x$. Find the domain and range of those relations which are functions.



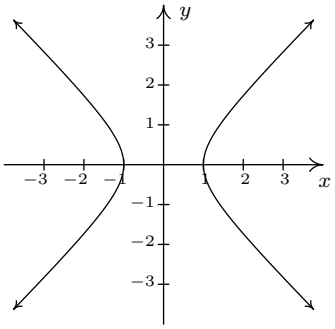

(a)



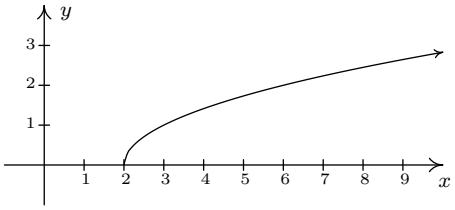

(b)



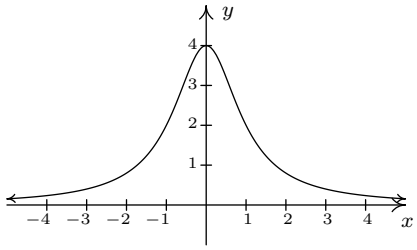

(c)



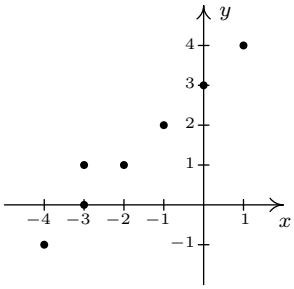

(d)



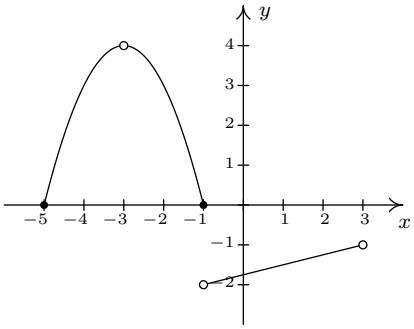

(e)



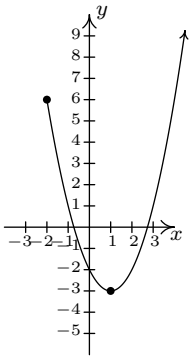

(f)



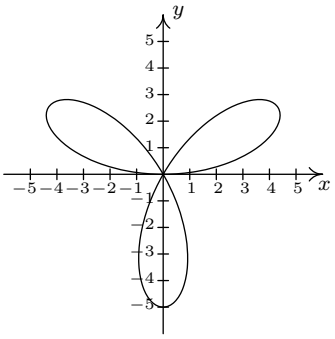

(g)



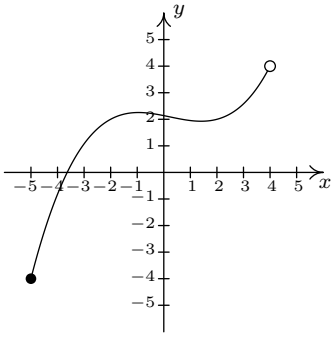

(h)



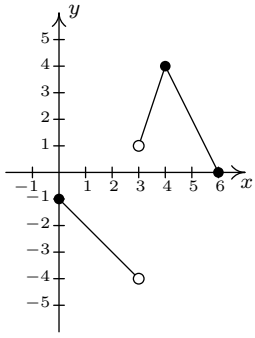

(i)




(j)




(k)




(l)

3. Determine which of the following equations represent $y$ as a function of $x$.

   (a) $y = x^3 - x$
   (b) $y = \sqrt{x-2}$
   (c) $x^3 y = -4$
   (d) $x^2 - y^2 = 1$
   (e) $y = \dfrac{x}{x^2 - 9}$
   (f) $x = -6$
   (g) $x = y^2 + 4$
   (h) $y = x^2 + 4$
   (i) $x^2 + y^2 = 4$
   (j) $y = \sqrt{4 - x^2}$
   (k) $x^2 - y^2 = 4$
   (l) $x^3 + y^3 = 4$
   (m) $2x + 3y = 4$
   (n) $2xy = 4$

4. Explain why the height $h$ of a Sasquatch is a function of its age $N$ in years. Given that a Sasquatch is 2 feet tall at birth, experiences growth spurts at ages 3, 23 and 57, and lives to be about 150 years old with a maximum height of 9 feet, sketch a rough graph of the height function.

5. Explain why the population $P$ of Sasquatch in a given area is a function of time $t$. What would be the range of this function?

6. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

7. The process given in Example 2.1.5 for determining whether an equation of a relation represents $y$ as a function of $x$ breaks down if we cannot solve the equation for $y$ in terms of $x$. However, that does not prevent us from proving that an equation which fails to represent $y$ as a function of $x$ actually fails to do so. What we really need is two points with the same $x$-coordinate and different $y$-coordinates which both satisfy the equation so that the graph of the relation would fail the Vertical Line Test 2.1. Discuss with your classmates how you might find such points for the relations given below.

(a) $x^3 + y^3 - 3xy = 0$
(b) $x^4 = x^2 + y^2$
(c) $y^2 = x^3 + 3x^2$
(d) $(x^2 + y^2)^2 = x^3 + y^3$

### 2.1.2 ANSWERS

1. (a) Function, domain $= \{-3, -2, -1, 0, 1, 2, 3\}$, range $= \{0, 1, 4, 9\}$.
(b) Not a function.
(c) Function, domain $= \{-7, -3, 3, 4, 5, 6\}$, range $= \{0, 4, 5, 6, 9\}$
(d) Function, domain $= \{1, 4, 9, 16, 25, 36, \ldots\} = \{x : x \text{ is a perfect square}\}$, range $= \{2, 4, 6, 8, 10, 12, \ldots\} = \{y : y \text{ is a positive even integer}\}$
(e) Not a function
(f) Function, domain $= \{x : x \text{ is irrational}\}$, range $= \{1\}$.
(g) Function, domain $= \{x : x = 2^n \text{ for some whole number } n\}$, range $= \{y : y \text{ is any whole number}\}$,
(h) Function, domain $= \{x : x \text{ is any integer}\}$, range $= \{y : y = n^2 \text{ for some integer } n\}$.
(i) Not a function.
(j) Function, domain $= [-2, 4)$, range $= \{3\}$.

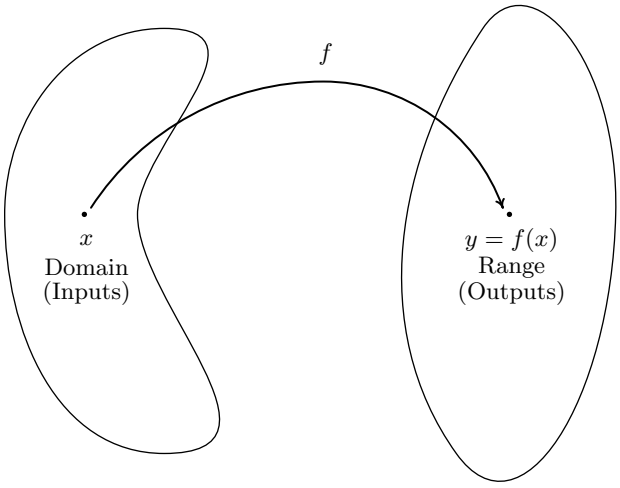
2. (a) Function, domain $= \{-4, -3, -2, -1, 0, 1\}$, range $= \{-1, 0, 1, 2, 3, 4\}$
(b) Not a function
(c) Function, domain $= (-\infty, \infty)$, range $= [1, \infty)$
(d) Not a function
(e) Function, domain $= [2, \infty)$, range $= [0, \infty)$
(f) Function, domain $= (-\infty, \infty)$, range $= (0, 4]$
(g) Not a function
(h) Function, domain $= [-5, -3) \cup (-3, 3)$, range $= (-2, -1) \cup [0, 4)$
(i) Function, domain $= [-2, \infty)$, range $= [-3, \infty)$
(j) Not a function
(k) Function, domain $= [-5, 4)$, range $= [-4, 4)$
(l) Function, domain $= [0, 3) \cup (3, 6]$, range $= (-4, -1] \cup [0, 4]$

3. (a) Function
(b) Function
(c) Function
(d) Not a function
(e) Function
(f) Not a function
(g) Not a function
(h) Function

(i) Not a function
(j) Function
(k) Not a function
(l) Function
(m) Function
(n) Function

## 2.2 Function Notation

In Definition 2.1, we described a function as a special kind of relation – one in which each $x$-coordinate is matched with only one $y$-coordinate. In this section, we focus more on the **process** by which the $x$ is matched with the $y$. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function $f$ as a process by which each input $x$ is matched with only one output $y$. Since the output is completely determined by the input $x$ and the process $f$, we symbolize the output with **function notation**: '$f(x)$', read '$f$ **of** $x$.' In this case, the parentheses here do not indicate multiplication, as they do elsewhere in algebra. This could cause confusion if the context is not clear. In other words, $f(x)$ is the **output** which results by applying the **process** $f$ to the **input** $x$. This relationship is typically visualized using a diagram similar to the one below.




The value of $y$ is completely dependent on the choice of $x$. For this reason, $x$ is often called the **independent variable**, or **argument** of $f$, whereas $y$ is often called the **dependent variable**.

As we shall see, the process of a function $f$ is usually described using an algebraic formula. For example, suppose a function $f$ takes a real number and performs the following two steps, in sequence

1. multiply by 3
2. add 4

If we choose 5 as our input, in step 1 we multiply by 3 to get $(5)(3) = 15$. In step 2, we add 4 to our result from step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process $f$ to the input 5 gives the output 19. In general, if we use $x$ for the input, applying step 1 produces $3x$. Following with step 2 produces $3x + 4$ as our final output. Hence for an input $x$, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we **replace** the occurrence of $x$ in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

EXAMPLE 2.2.1. Suppose a function $g$ is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

SOLUTION. Starting with 5, step 1 gives $5 + 4 = 9$. Continuing with step 2, we get $(3)(9) = 27$. To find a formula for $g(x)$, we start with our input $x$. Step 1 produces $x + 4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x + 4) = 3x + 12$. Hence, $g(x) = 3x + 12$. We can check our formula by replacing $x$ with 5 to get $g(5) = 3(5) + 12 = 15 + 12 = 27 \checkmark$. □

Most of the functions we will encounter in College Algebra will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other math courses.

EXAMPLE 2.2.2. For $f(x) = -x^2 + 3x + 4$, find and simplify

1. $f(-1)$, $f(0)$, $f(2)$
2. $f(2x)$, $2f(x)$
3. $f(x + 2)$, $f(x) + 2$, $f(x) + f(2)$

SOLUTION.

1. To find $f(-1)$, we replace every occurrence of $x$ in the expression $f(x)$ with $-1$

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, $f(0) = -(0)^2 + 3(0) + 4 = 4$, and $f(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

2. To find $f(2x)$, we replace every occurrence of $x$ with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

Note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the **input** by 2; for $2f(x)$, we are multiplying the **output** by 2. As we see, we get entirely different results. Also note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

3. To find $f(x+2)$, we replace every occurrence of $x$ with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -\left(x^2 + 4x + 4\right) + (3x + 6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find $f(x) + 2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x) + 2 &= \left(-x^2 + 3x + 4\right) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

Once again, we see there is a dramatic difference between modifying the input and modifying the output. Finally, in $f(x) + f(2)$ we are adding the value $f(2)$ to the expression $f(x)$. From our work above, we see $f(2) = 6$ so that

$$\begin{aligned} f(x) + f(2) &= \left(-x^2 + 3x + 4\right) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

Notice that $f(x+2)$, $f(x) + 2$ and $f(x) + f(2)$ are three **different** expressions. Even though function notation uses parentheses, as does multiplication, there is no general 'distributive property' of function notation. □

Suppose we wish to find $r(3)$ for $r(x) = \dfrac{2x}{x^2 - 9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2 - 9} = \frac{6}{0},$$

which is undefined. The number 3 is not an allowable input to the function $r$; in other words, 3 is not in the domain of $r$. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and $-3$, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume (unless the contrary is explicitly stated) that the domain of the function is the set of all real numbers for which the formula makes arithmetic sense when the number is substituted into the formula. This set of numbers is often called the **implied domain**[1] of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

EXAMPLE 2.2.3. Find the domain[2] of the following functions.

1. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x - 3}}$
2. $g(x) = \sqrt{4 - 3x}$
3. $h(x) = \sqrt[5]{4 - 3x}$
4. $r(x) = \dfrac{4}{6 - \sqrt{x + 3}}$
5. $I(x) = \dfrac{3x^2}{x}$

SOLUTION.

1. In the expression for $f$, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator,

[1] or, 'implicit domain'
[2] The word 'implied' is, well, implied.

we get $x - 3 = 0$ or $x = 3$. For the 'large' denominator

$$\begin{aligned} 1 - \frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\ x - 3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers **except** $-1$ and 3: $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

2. The potential disaster for $g$ is if the radicand[3] is negative. To avoid this, we set $4 - 3x \geq 0$

$$\begin{aligned} 4 - 3x &\geq 0 \\ 4 &\geq 3x \\ \frac{4}{3} &\geq x \end{aligned}$$

Hence, as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

3. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference – whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root.) Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to $h$. Hence, the domain is $(-\infty, \infty)$.

4. To find the domain of $r$, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a square root in that denominator. To satisfy the square

[3]The 'radicand' is the expression 'inside' the radical.

root, we set the radicand $x+3 \geq 0$ so $x \geq -3$. Setting the denominator equal to zero gives

$$\begin{aligned} 6-\sqrt{x+3} &= 0 \\ 6 &= \sqrt{x+3} \\ 6^2 &= \left(\sqrt{x+3}\right)^2 \\ 36 &= x+3 \\ 33 &= x \end{aligned}$$

Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $x = 33$, $6-\sqrt{x+3} = 6-\sqrt{36} = 0$, and so $x = 33$ will cause problems in the denominator. At last we can find the domain of $r$: we need $x \geq -3$, but $x \neq 33$. Our final answer is $[-3, 33) \cup (33, \infty)$.

5. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined.[4] Proceeding as before, we find the domain of $I$ to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$. □

It is worth reiterating the importance of finding the domain of a function **before** simplifying, as evidenced by the function $I$ in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. To understand why this is so important imagine writing a computer program to evaluate $I(x)$. The computer is told to input $x$, square it, multiply by 3 and divide by $x$. If the input is zero, the program will generate an error because division by zero is undefined. Most computer programs are not sophisticated enough to do the preliminary simplification.

| To find the domain of a function defined by an expression, do so before simplifying! |
|---|

Our next example shows how a function can be used to model real-world phenomena.

EXAMPLE 2.2.4. The height $h$ in feet of a model rocket above the ground $t$ seconds after lift off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } 0 \leq t \leq 20 \\ 0, & \text{if } t > 20 \end{cases}$$

Find and interpret $h(10)$ and $h(60)$.

[4] More precisely, the fraction $\frac{0}{0}$ is an 'indeterminant form'. Much time will be spent in Calculus wrestling with such creatures.

Solution. There are a few qualities of $h$ which may be off-putting. The first is that, unlike previous examples, the independent variable is $t$, not $x$. In this context, $t$ is chosen because it represents time. The second is that the function is broken up into two rules: one formula for values of $t$ between 0 and 20 inclusive, and another for values of $t$ greater than 20. To find $h(10)$, we first notice that 10 is between 0 and 20 so we use the first formula listed: $h(t) = -5t^2 + 100t$. Hence, $h(10) = -5(10)^2 + 100(10) = 500$. In terms of the model rocket, this means that 10 seconds after lift off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that 60 is greater than 20, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for $t$, so $h(60) = 0$. This means that 60 seconds after lift off, the rocket is 0 feet above the ground; in other words, a minute after lift off, the rocket has already returned to earth. □

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena (e.g. postal rates,[5] income tax formulas[6]) are modeled by such functions. Also note that the domain of $h$ in the above example is restricted to $t \geq 0$. For example, $h(-3)$ is not defined because $t = -3$ doesn't satisfy any of the conditions in any of the function's pieces. There is no inherent arithmetic reason which prevents us from calculating, say, $-5(-3)^2 + 100(-3)$, it's just that in this applied setting, $t = -3$ is meaningless. In this case, we say $h$ has an **applied domain**of $[0, \infty)$

### 2.2.1 Exercises

1. Suppose $f$ is a function that takes a real number $x$ and performs the following three steps in the order given: (1) square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4. Find an expression for $f(x)$ and find its domain.

2. Suppose $g$ is a function that takes a real number $x$ and performs the following three steps in the order given: (1) subtract 13; (2) square root; (3) make the quantity the denominator of a fraction with numerator 4. Find an expression for $g(x)$ and find its domain.

3. Suppose $h$ is a function that takes a real number $x$ and performs the following three steps in the order given: (1) square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13. Find an expression for $h(x)$ and find its domain.

4. Suppose $k$ is a function that takes a real number $x$ and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) square root; (3) subtract 13. Find an expression for $k(x)$ and find its domain.

5. For $f(x) = x^2 - 3x + 2$, find and simplify the following:

   (a) $f(3)$
   (b) $f(-1)$
   (c) $f\left(\frac{3}{2}\right)$
   (d) $f(4x)$
   (e) $4f(x)$
   (f) $f(-x)$
   (g) $f(x-4)$
   (h) $f(x) - 4$
   (i) $f\left(x^2\right)$

[5] See the United States Postal Service website http://www.usps.com/prices/first-class-mail-prices.htm
[6] See the Internal Revenue Service's website http://www.irs.gov/pub/irs-pdf/i1040tt.pdf

6. Repeat Exercise 5 above for $f(x) = \dfrac{2}{x^3}$

7. Let $f(x) = 3x^2 + 3x - 2$. Find and simplify the following:

   (a) $f(2)$
   (b) $f(-2)$
   (c) $f(2a)$
   (d) $2f(a)$
   (e) $f(a+2)$
   (f) $f(a) + f(2)$
   (g) $f\left(\frac{2}{a}\right)$
   (h) $\frac{f(a)}{2}$
   (i) $f(a+h)$

8. Let $f(x) = \begin{cases} x+5, & x \leq -3 \\ \sqrt{9-x^2}, & -3 < x \leq 3 \\ -x+5, & x > 3 \end{cases}$

   (a) $f(-4)$
   (b) $f(-3)$
   (c) $f(3)$
   (d) $f(3.001)$
   (e) $f(-3.001)$
   (f) $f(2)$

9. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \leq 1 \\ x & \text{if} \quad x > 1 \end{cases}$ Compute the following function values.

   (a) $f(4)$
   (b) $f(-3)$
   (c) $f(1)$
   (d) $f(0)$
   (e) $f(-1)$
   (f) $f(-0.999)$

10. Find the (implied) domain of the function.

    (a) $f(x) = x^4 - 13x^3 + 56x^2 - 19$
    (b) $f(x) = x^2 + 4$
    (c) $f(x) = \dfrac{x+4}{x^2-36}$
    (d) $f(x) = \sqrt{6x-2}$
    (e) $f(x) = \dfrac{6}{\sqrt{6x-2}}$
    (f) $f(x) = \sqrt[3]{6x-2}$
    (g) $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$
    (h) $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
    (i) $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$
    (j) $s(t) = \dfrac{t}{t-8}$
    (k) $Q(r) = \dfrac{\sqrt{r}}{r-8}$
    (l) $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$
    (m) $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$

(n) $A(x) = \sqrt{x-7} + \sqrt{9-x}$

(o) $g(v) = \dfrac{1}{4 - \dfrac{1}{v^2}}$

(p) $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

11. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \dfrac{150t}{t+15}$, where $t = 0$ represents the year 1803. What is the applied domain of $P$? What range "makes sense" for this function? What does $P(0)$ represent? What does $P(205)$ represent?

12. Recall that the set of **integers** is the set of numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[7] The **greatest integer of** $x$, $\lfloor x \rfloor$, is defined to be the largest integer $k$ with $k \leq x$.

    (a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$

    (b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piece-wise defined function.

    **HINT:** There are infinitely many pieces!

    (c) Is $\lfloor a+b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if $a$ or $b$ is an integer? Test some values, make a conjecture, and explain your result.

13. Graph the following function:[8]

| If your income is | over | but not over | your 2009 tax is | of the amount over |
|---|---|---|---|---|
| | \$0 | \$13,620 | 4.60% | \$0 |
| | \$13,620 | \$27,250 | \$626.52 + 6.15% | \$13,620 |
| | \$27,250 | \$204,370 | \$1,464.77 + 6.50% | \$27,250 |
| | \$204,370 | \$300,000 | \$12,977.57 + 6.75% | \$204,370 |
| | \$300,000 | | \$19,432.60 + 7.75% | \$300,000 |

14. Graph the following function:[9]

[7] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count.'

[8] http://www.revenue.wi.gov/faqs/pcs/taxrates.html

[9] http://www.irs.gov/pub/irs-pdf/i1040tt.pdf

| If your taxable income is | over | but not over | your tax is | of the amount over |
|---|---|---|---|---|
| | \$0 | \$16,700.00 | ... 10% | \$0 |
| | 16,700 | 67,900.00 | 1,670 + 15% | 16,700 |
| | 67,900 | 137,050 .50 | 9,350 + 25% | 67,900 |
| | 137,050 | 208,850.50 | 26,637 + 28% | 137,050 |
| | 208,850 | 372,950.50 | 46,741 + 33% | 208,850 |
| | 372,950 | ... | 100,895 + 35% | 372,950 |

15. We have through our examples tried to convince you that, in general, $f(a+b) \neq f(a) + f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function $f$ for which the following properties are always true.

    (a) $f(0) = f(-1+1) = f(-1) + f(1)$
    (b) $f(5) = f(2+3) = f(2) + f(3)$
    (c) $f(-6) = f(0-6) = f(0) - f(6)$
    (d) $f(a+b) = f(a) + f(b)$ regardless of what two numbers we give you for $a$ and $b$.

    How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \dfrac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a+b) \neq f(a) + f(b)$.

### 2.2.2 ANSWERS

1. $f(x) = \dfrac{4}{\sqrt{x} - 13}$
   Domain: $[0, 169) \cup (169, \infty)$

2. $g(x) = \dfrac{4}{\sqrt{x - 13}}$
   Domain: $(13, \infty)$

3. $h(x) = \dfrac{4}{\sqrt{x}} - 13$
   Domain: $(0, \infty)$

4. $k(x) = \sqrt{\dfrac{4}{x}} - 13$
   Domain: $(0, \infty)$

5. (a) 2
   (b) 6
   (c) $-\dfrac{1}{4}$
   (d) $16x^2 - 12x + 2$
   (e) $4x^2 - 12x + 8$
   (f) $x^2 + 3x + 2$
   (g) $x^2 - 11x + 30$
   (h) $x^2 - 3x - 2$
   (i) $x^4 - 3x^2 + 2$

6. (a) $\dfrac{2}{27}$
   (b) $-2$
   (c) $\dfrac{16}{27}$
   (d) $\dfrac{1}{32x^3}$
   (e) $\dfrac{8}{x^3}$
   (f) $-\dfrac{2}{x^3}$
   (g) $\dfrac{2}{(x-4)^3} = \dfrac{2}{x^3 - 12x^2 + 48x - 64}$
   (h) $\dfrac{2}{x^3} - 4 = \dfrac{2 - 4x^3}{x^3}$
   (i) $\dfrac{2}{x^6}$

7. (a) 16
   (b) 4
   (c) $12a^2 + 6a - 2$
   (d) $6a^2 + 6a - 4$
   (e) $3a^2 + 15a + 14$
   (f) $3a^2 + 3a + 14$
   (g) $\frac{12}{a^2} + \frac{6}{a} - 2$
   (h) $\frac{3a^2}{2} + \frac{3a}{2} - 1$
   (i) $3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

8. (a) $f(-4) = 1$
   (b) $f(-3) = 2$
   (c) $f(3) = 0$
   (d) $f(3.001) = 1.999$
   (e) $f(-3.001) = 1.999$
   (f) $f(2) = \sqrt{5}$

9. (a) $f(4) = 4$
   (b) $f(-3) = 9$
   (c) $f(1) = 0$
   (d) $f(0) = 1$
   (e) $f(-1) = 1$
   (f) $f(-0.999) \approx 0.0447101778$

10. (a) $(-\infty, \infty)$
    (b) $(-\infty, \infty)$
    (c) $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
    (d) $\left[\frac{1}{3}, \infty\right)$
    (e) $\left(\frac{1}{3}, \infty\right)$
    (f) $(-\infty, \infty)$
    (g) $\left[\frac{1}{3}, 3\right) \cup (3, \infty)$
    (h) $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$
    (i) $(-\infty, \infty)$
    (j) $(-\infty, 8) \cup (8, \infty)$
    (k) $[0, 8) \cup (8, \infty)$
    (l) $(8, \infty)$
    (m) $(-\infty, 8) \cup (8, \infty)$
    (n) $[7, 9]$
    (o) $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
    (p) $[0, 25) \cup (25, \infty)$

11. The applied domain of $P$ is $[0, \infty)$. The range is some subset of the natural numbers because we cannot have fractional Sasquatch. This was a bit of a trick question and we'll address the notion of mathematical modeling more thoroughly in later chapters. $P(0) = 0$ means that there were no Sasquatch in Portage County in 1803. $P(205) \approx 139.77$ would mean there were 139 or 140 Sasquatch in Portage County in 2008.

12. (a) $\lfloor 0.785 \rfloor = 0$, $\lfloor 117 \rfloor = 117$, $\lfloor -2.001 \rfloor = -3$, and $\lfloor \pi + 6 \rfloor = 9$

## 2.3 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x) + 2$' and '$2f(x)$' for a given function $f$. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

**Function Arithmetic**

Suppose $f$ and $g$ are functions and $x$ is an element common to the domains of $f$ and $g$.

- The **sum** of $f$ and $g$, denoted $f + g$, is the function defined by the formula:

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of $f$ and $g$, denoted $f - g$, is the function defined by the formula:

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of $f$ and $g$, denoted $fg$, is the function defined by the formula:

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of $f$ and $g$, denoted $\dfrac{f}{g}$, is the function defined by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided $g(x) \neq 0$.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on.

EXAMPLE 2.3.1. Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$. Find and simplify expressions for for the following functions. In addition, find the domain of each of these functions.

1. $(f + g)(x)$
2. $(g - f)(x)$
3. $(fg)(x)$
4. $\left(\dfrac{g}{f}\right)(x)$

Solution.

1. $(f+g)(x)$ is defined to be $f(x)+g(x)$. To that end, we get

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= \left(6x^2-2x\right)+\left(3-\frac{1}{x}\right)\\ &= 6x^2-2x+3-\frac{1}{x}\\ &= \frac{6x^3}{x}-\frac{2x^2}{x}+\frac{3x}{x}-\frac{1}{x} \quad \text{get common denominators}\\ &= \frac{6x^3-2x^2+3x-1}{x}\end{aligned}$$

To find the domain of $(f+g)$ we do so **before** we simplify, that is, at the step

$$\left(6x^2-2x\right)+\left(3-\frac{1}{x}\right)$$

We see $x \neq 0$, but everything else is fine. Hence, the domain is $(-\infty, 0) \cup (0, \infty)$.

2. $(g-f)(x)$ is defined to be $g(x)-f(x)$. To that end, we get

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)\\ &= 3-\frac{1}{x}-6x^2+2x\\ &= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x} \quad \text{get common denominators}\\ &= \frac{-6x^3+2x^2+3x-1}{x}\end{aligned}$$

Looking at the expression for $(g - f)$ before we simplified

$$\left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right)$$

we see, as before, $x \neq 0$ is the only restriction. The domain is $(-\infty, 0) \cup (0, \infty)$.

3. $(fg)(x)$ is defined to be $f(x)g(x)$. Substituting yields

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right) \\
&= \left(6x^2 - 2x\right)\left(\frac{3x - 1}{x}\right) \\
&= \left(\frac{2x(3x - 1)}{1}\right)\left(\frac{3x - 1}{x}\right) \quad \text{factor} \\
&= \left(\frac{2\cancel{x}(3x - 1)}{1}\right)\left(\frac{3x - 1}{\cancel{x}}\right) \quad \text{cancel} \\
&= 2(3x - 1)^2 \\
&= 2\left(9x^2 - 6x + 1\right) \\
&= 18x^2 - 12x + 2
\end{aligned}$$

To determine the domain, we check the step just after we substituted

$$\left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right)$$

which gives us, as before, the domain: $(-\infty, 0) \cup (0, \infty)$.

4. $\left(\frac{g}{f}\right)(x)$ is defined to be $\frac{g(x)}{f(x)}$. Thus we have

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x}\cdot\frac{x}{x} && \text{simplify complex fractions} \\
&= \frac{\left(3-\frac{1}{x}\right)x}{(6x^2-2x)\,x} \\
&= \frac{3x-1}{(6x^2-2x)\,x} \\
&= \frac{3x-1}{2x^2(3x-1)} && \text{factor} \\
&= \frac{\cancel{(3x-1)}^{\,1}}{2x^2\cancel{(3x-1)}} && \text{cancel} \\
&= \frac{1}{2x^2}
\end{aligned}$$

To find the domain, we consider the first step after substitution:

$$\frac{3-\frac{1}{x}}{6x^2-2x}$$

To avoid division by zero in the 'little' fraction, $\frac{1}{x}$, we need $x \neq 0$. For the 'big' fraction we set $6x^2-2x=0$ and solve: $2x(3x-1)=0$ and get $x=0,\frac{1}{3}$. Thus we must exclude $x=\frac{1}{3}$ as well, resulting in a domain of $(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$. □

We close this section with concept of the **difference quotient** of a function. It is a critical tool for Calculus and also a great way to practice function notation.[1]

[1] You may need to brush up on your Intermediate Algebra skills, as well.

DEFINITION 2.3. Given a function, $f$, the **difference quotient** of $f$ is the expression:

$$\frac{f(x+h)-f(x)}{h}$$

EXAMPLE 2.3.2. Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$

SOLUTION.

1. To find $f(x+h)$, we replace every occurrence of $x$ in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\ &= \frac{2xh+h^2-h}{h} \\ &= \frac{h\,(2x+h-1)}{h} && \text{factor} \\ &= \frac{\cancel{h}\,(2x+h-1)}{\cancel{h}} && \text{cancel} \\ &= 2x+h-1. \end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of $x$ in the formula $g(x) = \dfrac{3}{2x+1}$ with the quantity $(x+h)$

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{\dfrac{3}{2x+2h+1} - \dfrac{3}{2x+1}}{h} \\ &= \frac{\dfrac{3}{2x+2h+1} - \dfrac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\ &= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\ &= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6h}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \\ &= \frac{-6}{(2x+2h+1)(2x+1)}. \end{aligned}$$

For reasons which will become clear in Calculus, we do not expand the denominator. □

### 2.3.1 Exercises

1. Let $f(x) = \sqrt{x}$, $g(x) = x+10$ and $h(x) = \dfrac{1}{x}$.

   (a) Compute the following function values.

i. $(f+g)(4)$ ii. $(g-h)(7)$ iii. $(fh)(25)$ iv. $\left(\frac{h}{g}\right)(3)$

(b) Find the domain of the following functions then simplify their expressions.

i. $(f+g)(x)$ iii. $(fh)(x)$ v. $\left(\frac{g}{h}\right)(x)$

ii. $(g-h)(x)$ iv. $\left(\frac{h}{g}\right)(x)$ vi. $(h-f)(x)$

2. Let $f(x) = 3\sqrt{x} - 1$, $g(x) = 2x^2 - 3x - 2$ and $h(x) = \dfrac{3}{2-x}$.

(a) Compute the following function values.

i. $(f+g)(4)$ ii. $(g-h)(1)$ iii. $(fh)(0)$ iv. $\left(\frac{h}{g}\right)(-1)$

(b) Find the domain of the following functions then simplify their expressions.

i. $(f-g)(x)$ ii. $(gh)(x)$ iii. $\left(\frac{f}{g}\right)(x)$ iv. $\left(\frac{f}{h}\right)(x)$

3. Let $f(x) = \sqrt{6x-2}$, $g(x) = x^2 - 36$, and $h(x) = \dfrac{1}{x-4}$.

(a) Compute the following function values.

i. $(f+g)(3)$ iii. $\left(\frac{f}{g}\right)(4)$ v. $(g+h)(-4)$

ii. $(g-h)(8)$ iv. $(fh)(8)$ vi. $\left(\frac{h}{g}\right)(-12)$

(b) Find the domain of the following functions and simplify their expressions.

i. $(f+g)(x)$ iii. $\left(\frac{f}{g}\right)(x)$ v. $(g+h)(x)$

ii. $(g-h)(x)$ iv. $(fh)(x)$ vi. $\left(\frac{h}{g}\right)(x)$

4. Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the following functions.

(a) $f(x) = 2x - 5$

(b) $f(x) = -3x + 5$

(c) $f(x) = 6$

(d) $f(x) = 3x^2 - x$

(e) $f(x) = -x^2 + 2x - 1$

(f) $f(x) = x^3 + 1$

(g) $f(x) = \dfrac{2}{x}$

(h) $f(x) = \dfrac{3}{1-x}$

(i) $f(x) = \dfrac{x}{x-9}$

(j) $f(x) = \sqrt{x}$ [2]

(k) $f(x) = mx + b$ where $m \neq 0$

(l) $f(x) = ax^2 + bx + c$ where $a \neq 0$

### 2.3.2 Answers

1. (a) i. $(f+g)(4) = 16$ ii. $(g-h)(7) = \frac{118}{7}$ iii. $(fh)(25) = \frac{1}{5}$ iv. $\left(\frac{h}{g}\right)(3) = \frac{1}{39}$

   (b) i. $(f+g)(x) = \sqrt{x} + x + 10$
   Domain: $[0, \infty)$

   ii. $(g-h)(x) = x + 10 - \frac{1}{x}$
   Domain: $(-\infty, 0) \cup (0, \infty)$

   iii. $(fh)(x) = \frac{1}{\sqrt{x}}$
   Domain: $(0, \infty)$

   iv. $\left(\frac{h}{g}\right)(x) = \frac{1}{x(x+10)}$
   Domain: $(-\infty, -10) \cup (-10, 0) \cup (0, \infty)$

   v. $\left(\frac{g}{h}\right)(x) = x(x+10)$
   Domain: $(-\infty, 0) \cup (0, \infty)$

   vi. $(h-f)(x) = \frac{1}{x} - \sqrt{x}$
   Domain: $(0, \infty)$

2. (a) i. $(f+g)(4) = 23$ ii. $(g-h)(1) = -6$ iii. $(fh)(0) = -\frac{3}{2}$ iv. $\left(\frac{h}{g}\right)(-1) = \frac{1}{3}$

   (b) i. $(f-g)(x) = -2x^2 + 3x + 3\sqrt{x} + 1$
   Domain: $[0, \infty)$

   ii. $(gh)(x) = -6x - 3$
   Domain: $(-\infty, 2) \cup (2, \infty)$

   iii. $\left(\frac{f}{g}\right)(x) = \frac{3\sqrt{x} - 1}{2x^2 - 3x - 2}$
   Domain: $[0, 2) \cup (2, \infty)$

   iv. $\left(\frac{f}{h}\right)(x) = -x\sqrt{x} + \frac{1}{3}x + 2\sqrt{x} - \frac{2}{3}$
   Domain: $[0, 2) \cup (2, \infty)$

3. (a) i. $(f+g)(3) = -23$ iii. $\left(\frac{f}{g}\right)(4) = -\frac{\sqrt{22}}{20}$ v. $(g+h)(-4) = -\frac{161}{8}$

   ii. $(g-h)(8) = \frac{111}{4}$ iv. $(fh)(8) = \frac{\sqrt{46}}{4}$ vi. $\left(\frac{h}{g}\right)(-12) = -\frac{1}{1728}$

   (b) i. $(f+g)(x) = x^2 - 36 + \sqrt{6x-2}$
   Domain: $\left[\frac{1}{3}, \infty\right)$

   ii. $(g-h)(x) = x^2 - 36 - \frac{1}{x-4}$
   Domain: $(-\infty, 4) \cup (4, \infty)$

   iii. $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{6x-2}}{x^2 - 36}$
   Domain: $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

   iv. $(fh)(x) = \frac{\sqrt{6x-2}}{x-4}$
   Domain: $\left[\frac{1}{3}, 4\right) \cup (4, \infty)$

   v. $(g+h)(x) = x^2 - 36 + \frac{1}{x-4}$

[2] Rationalize the numerator. It won't look 'simplified' per se, but work through until you can cancel the 'h'.

Domain: $(-\infty, 4) \cup (4, \infty)$

vi. $\left(\frac{h}{g}\right)(x) = \frac{1}{(x-4)(x^2-36)}$

Domain: $(-\infty, -6) \cup (-6, 4) \cup (4, 6) \cup (6, \infty)$

4. (a) 2
   (b) $-3$
   (c) 0
   (d) $6x + 3h - 1$
   (e) $-2x - h + 2$
   (f) $3x^2 + 3xh + h^2$
   (g) $-\frac{2}{x(x+h)}$
   (h) $\frac{3}{(1-x-h)(1-x)}$
   (i) $\frac{-9}{(x-9)(x+h-9)}$
   (j) $\frac{1}{\sqrt{x+h}+\sqrt{x}}$
   (k) $m$
   (l) $2ax + ah + b$

## 2.4 GRAPHS OF FUNCTIONS

In Section 2.1 we defined a function as a special type of relation; one in which each $x$-coordinate was matched with only one $y$-coordinate. We spent most of our time in that section looking at functions graphically because they were, after all, just sets of points in the plane. Then in Section 2.2 we described a function as a process and defined the notation necessary to work with functions algebraically. So now it's time to look at functions graphically again, only this time we'll do so with the notation defined in Section 2.2. We start with what should not be a surprising connection.

**The Fundamental Graphing Principle for Functions**

The graph of a function $f$ is the set of points which satisfy the equation $y = f(x)$. That is, the point $(x, y)$ is on the graph of $f$ if and only if $y = f(x)$.

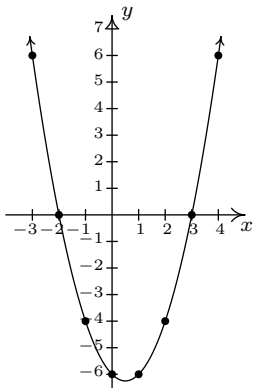
EXAMPLE 2.4.1. Graph $f(x) = x^2 - x - 6$.

SOLUTION. To graph $f$, we graph the equation $y = f(x)$. To this end, we use the techniques outlined in Section 1.3. Specifically, we check for intercepts, test for symmetry, and plot additional points as needed. To find the $x$-intercepts, we set $y = 0$. Since $y = f(x)$, this means $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2 - x - 6 \\ 0 &= x^2 - x - 6 \\ 0 &= (x-3)(x+2) \quad \text{factor} \\ x - 3 = 0 \quad &\text{or} \quad x + 2 = 0 \\ x &= -2, 3 \end{aligned}$$

So we get $(-2, 0)$ and $(3, 0)$ as $x$-intercepts. To find the $y$-intercept, we set $x = 0$. Using function notation, this is the same as finding $f(0)$ and $f(0) = 0^2 - 0 - 6 = -6$. Thus the $y$-intercept is $(0, -6)$. As far as symmetry is concerned, we can tell from the intercepts that the graph possesses none of the three symmetries discussed thus far. (You should verify this.) We can make a table analogous to the ones we made in Section 1.3, plot the points and connect the dots in a somewhat pleasing fashion to get the graph below on the right.

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-3$ | 6 | $(-3, 6)$ |
| $-2$ | 0 | $(-2, 0)$ |
| $-1$ | $-4$ | $(-1, -4)$ |
| 0 | $-6$ | $(0, -6)$ |
| 1 | $-6$ | $(1, -6)$ |
| 2 | $-4$ | $(2, -4)$ |
| 3 | 0 | $(3, 0)$ |
| 4 | 6 | $(4, 6)$ |




□

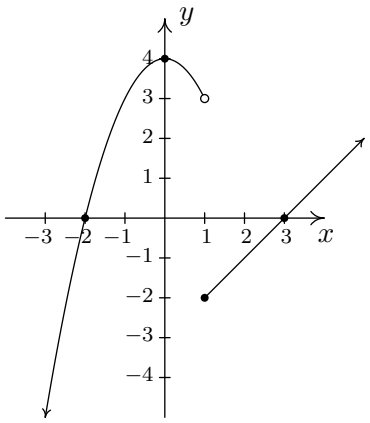
Graphing piecewise-defined functions is a bit more of a challenge.

Example 2.4.2. Graph: $f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x < 1 \\ x - 3, & \text{if} \quad x \geq 1 \end{cases}$

Solution. We proceed as before: finding intercepts, testing for symmetry and then plotting additional points as needed. To find the $x$-intercepts, as before, we set $f(x) = 0$. The twist is that we have two formulas for $f(x)$. For $x < 1$, we use the formula $f(x) = 4 - x^2$. Setting $f(x) = 0$ gives $0 = 4 - x^2$, so that $x = \pm 2$. However, of these two answers, only $x = -2$ fits in the domain $x < 1$ for this piece. This means the only $x$-intercept for the $x < 1$ region of the $x$-axis is $(-2, 0)$. For $x \geq 1$, $f(x) = x - 3$. Setting $f(x) = 0$ gives $0 = x - 3$, or $x = 3$. Since $x = 3$ satisfies the inequality $x \geq 1$, we get $(3, 0)$ as another $x$-intercept. Next, we seek the $y$-intercept. Notice that $x = 0$ falls in the domain $x < 1$. Thus $f(0) = 4 - 0^2 = 4$ yields the $y$-intercept $(0, 4)$. As far as symmetry is concerned, you can check that the equation $y = 4 - x^2$ is symmetric about the $y$-axis; unfortunately, this equation (and its symmetry) is valid only for $x < 1$. You can also verify

$y = x - 3$ possesses none of the symmetries discussed in the Section 1.3. When plotting additional points, it is important to keep in mind the restrictions on $x$ for each piece of the function. The sticking point for this function is $x = 1$, since this is where the equations change. When $x = 1$, we use the formula $f(x) = x - 3$, so the point on the graph $(1, f(1))$ is $(1, -2)$. However, for all values less than 1, we use the formula $f(x) = 4 - x^2$. As we have discussed earlier in Section 1.2, there is no real number which immediately precedes $x = 1$ on the number line. Thus for the values $x = 0.9$, $x = 0.99$, $x = 0.999$, and so on, we find the corresponding $y$ values using the formula $f(x) = 4 - x^2$. Making a table as before, we see that as the $x$ values sneak up to $x = 1$ in this fashion, the $f(x)$ values inch closer and closer[1] to $4 - 1^2 = 3$. To indicate this graphically, we use an open circle at the point $(1, 3)$. Putting all of this information together and plotting additional points, we get

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| 0.9 | 3.19 | $(0.9, 3.19)$ |
| 0.99 | $\approx 3.02$ | $(0.99, 3.02)$ |
| 0.999 | $\approx 3.002$ | $(0.999, 3.002)$ |




□

In the previous two examples, the $x$-coordinates of the $x$-intercepts of the graph of $y = f(x)$ were found by solving $f(x) = 0$. For this reason, they are called the **zeros** of $f$.

DEFINITION 2.4. The **zeros** of a function $f$ are the solutions to the equation $f(x) = 0$. In other words, $x$ is a zero of $f$ if and only if $(x, 0)$ is an $x$-intercept of the graph of $y = f(x)$.

Of the three symmetries discussed in Section 1.3, only two are of significance to functions: symmetry about the $y$-axis and symmetry about the origin.[2] Recall that we can test whether the graph of an equation is symmetric about the $y$-axis by replacing $x$ with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for $x$ results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. In a similar fashion, we recall that to test an equation's graph for symmetry about the origin, we replace $x$ and $y$ with $-x$ and $-y$, respectively. Doing

[1] We've just stepped into Calculus here!

[2] Why are we so dismissive about symmetry about the $x$-axis for graphs of functions?

this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for $y$ gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. These results are summarized below.

**Steps for testing if the graph of a function possesses symmetry**

The graph of a function $f$ is symmetric:

- About the $y$-axis if and only if $f(-x) = f(x)$ for all $x$ in the domain of $f$.
- About the origin if and only if $f(-x) = -f(x)$ for all $x$ in the domain of $f$.

For reasons which won't become clear until we study polynomials, we call a function **even** if its graph is symmetric about the $y$-axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd,[3] functions fall into one of three distinct categories: even, odd, or neither even nor odd.

Example 2.4.3. Analytically determine if the following functions are even, odd, or neither even nor odd. Verify your result with a graphing calculator.

1. $f(x) = \dfrac{5}{2 - x^2}$
2. $g(x) = \dfrac{5x}{2 - x^2}$
3. $h(x) = \dfrac{5x}{2 - x^3}$
4. $i(x) = \dfrac{5x}{2x - x^3}$
5. $j(x) = x^2 - \dfrac{x}{100} - 1$

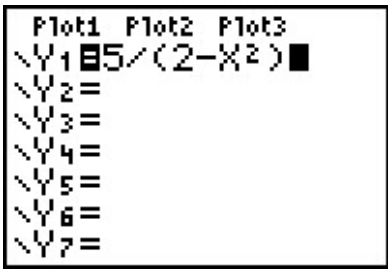
Solution. The first step in all of these problems is to replace $x$ with $-x$ and simplify.

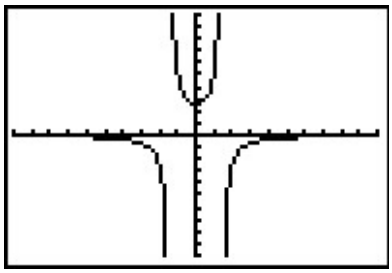
1.

$$\begin{aligned} f(x) &= \frac{5}{2 - x^2} \\ f(-x) &= \frac{5}{2 - (-x)^2} \\ f(-x) &= \frac{5}{2 - x^2} \\ f(-x) &= f(x) \end{aligned}$$

Hence, $f$ is **even**. The graphing calculator furnishes the following:

[3] Any ideas?



This suggests[4] the graph of $f$ is symmetric about the $y$-axis, as expected.

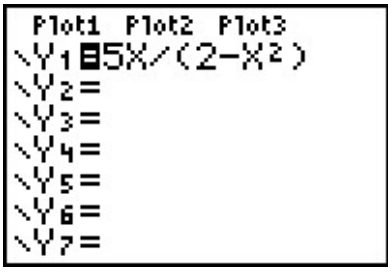
2.

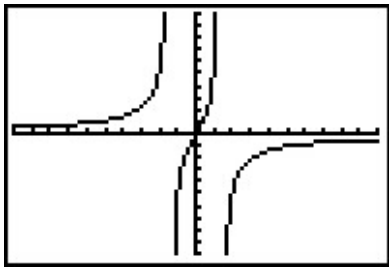
$$\begin{aligned} g(x) &= \frac{5x}{2-x^2} \\ g(-x) &= \frac{5(-x)}{2-(-x)^2} \\ g(-x) &= \frac{-5x}{2-x^2} \end{aligned}$$

It doesn't appear that $g(-x)$ is equivalent to $g(x)$. To prove this, we check with an $x$ value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that $g$ is not even, but it doesn't rule out the possibility that $g$ is odd. (Why not?) To check if $g$ is odd, we compare $g(-x)$ with $-g(x)$

$$\begin{aligned} -g(x) &= -\frac{5x}{2-x^2} \\ &= \frac{-5x}{2-x^2} \\ -g(x) &= g(-x) \end{aligned}$$

Hence, $g$ is odd. Graphically,



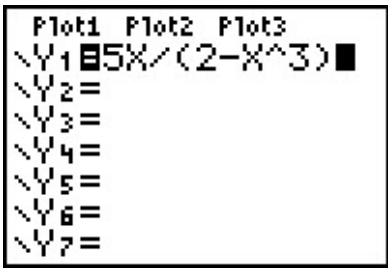
[4] 'Suggests' is about the extent of what a graphing calculator can do.

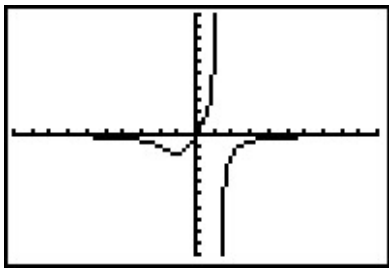
The calculator indicates the graph of $g$ is symmetric about the origin, as expected.

3.

$$\begin{aligned} h(x) &= \frac{5x}{2-x^3} \\ h(-x) &= \frac{5(-x)}{2-(-x)^3} \\ h(-x) &= \frac{-5x}{2+x^3} \end{aligned}$$

Once again, $h(-x)$ doesn't appear to be equivalent to $h(x)$. We check with an $x$ value, for example, $h(1) = 5$ but $h(-1) = -\frac{5}{3}$. This proves that $h$ is not even and it also shows $h$ is not odd. (Why?) Graphically,



The graph of $h$ appears to be neither symmetric about the $y$-axis nor the origin.

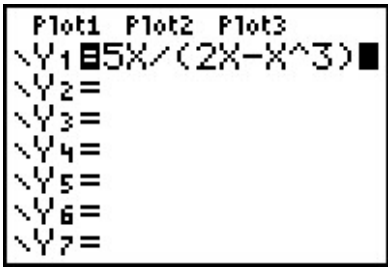
4.

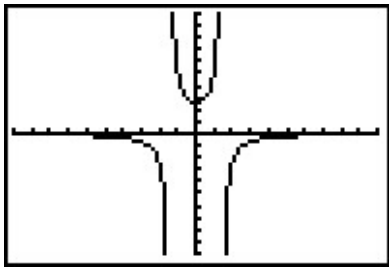
$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ i(-x) &= \frac{5(-x)}{2(-x)-(-x)^3} \\ i(-x) &= \frac{-5x}{-2x+x^3} \end{aligned}$$

The expression $i(-x)$ doesn't appear to be equivalent to $i(x)$. However, after checking some $x$ values, for example $x = 1$ yields $i(1) = 5$ and $i(-1) = 5$, it appears that $i(-x)$ does, in fact, equal $i(x)$. However, while this suggests $i$ is even, it doesn't **prove** it. (It does, however, prove $i$ is not odd.) To prove $i(-x) = i(x)$, we need to manipulate our expressions for $i(x)$ and $i(-x)$ and show they are equivalent. A clue as to how to proceed is in the numerators: in the formula for $i(x)$, the numerator is $5x$ and in $i(-x)$ the numerator is $-5x$. To re-write $i(x)$ with a numerator of $-5x$, we need to multiply its numerator by $-1$. To keep the value of the fraction the same, we need to multiply the denominator by $-1$ as well. Thus

$$\begin{aligned} i(x) &= \frac{5x}{2x - x^3} \\ &= \frac{(-1)5x}{(-1)(2x - x^3)} \\ &= \frac{-5x}{-2x + x^3} \end{aligned}$$

Hence, $i(x) = i(-x)$, so $i$ is even. The calculator supports our conclusion.



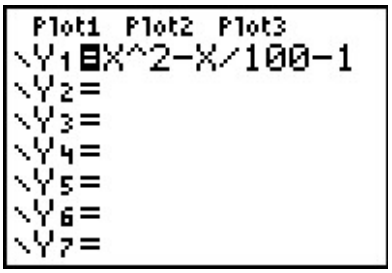
5.

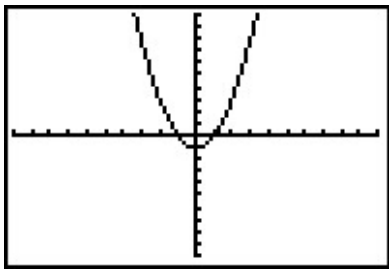
$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ j(-x) &= (-x)^2 - \frac{-x}{100} - 1 \\ j(-x) &= x^2 + \frac{x}{100} - 1 \end{aligned}$$

The expression for $j(-x)$ doesn't seem to be equivalent to $j(x)$, so we check using $x = 1$ to get $j(1) = -\frac{1}{100}$ and $j(-1) = \frac{1}{100}$. This rules out $j$ being even. However, it doesn't rule out $j$ being odd. Examining $-j(x)$ gives

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ -j(x) &= -\left(x^2 - \frac{x}{100} - 1\right) \\ -j(x) &= -x^2 + \frac{x}{100} + 1 \end{aligned}$$

The expression $-j(x)$ doesn't seem to match $j(-x)$ either. Testing $x = 2$ gives $j(2) = \frac{149}{50}$ and $j(-2) = \frac{151}{50}$, so $j$ is not odd, either. The calculator gives:



The calculator suggests that the graph of $j$ is symmetric about the $y$-axis which would imply that $j$ is even. However, we have proven that is not the case. □

There are two lessons to be learned from the last example. The first is that sampling function values at particular $x$ values is not enough to prove that a function is even or odd – despite the fact that $j(-1) = -j(1)$, $j$ turned out not to be odd. Secondly, while the calculator may **suggest** mathematical truths, it is the algebra which **proves** mathematical truths.[5]

### 2.4.1 General Function Behavior

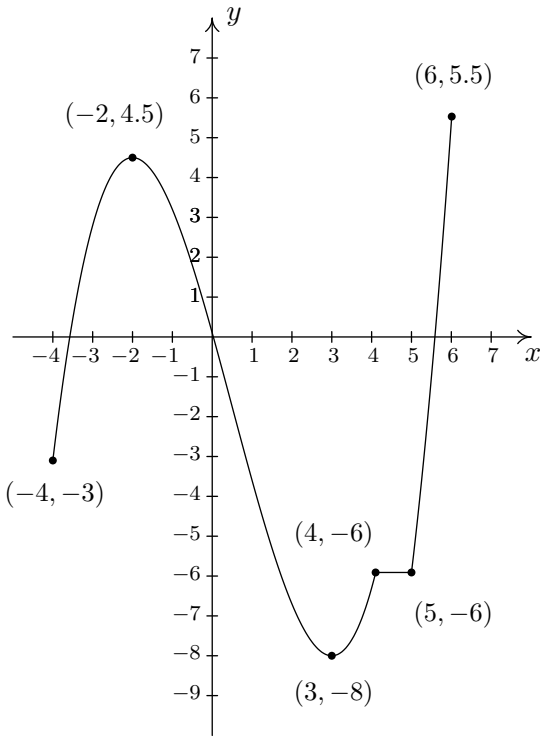
The last topic we wish to address in this section is general function behavior. As you shall see in the next several chapters, each family of functions has its own unique attributes and we will study them all in great detail. The purpose of this section's discussion, then, is to lay the foundation for that further study by investigating aspects of function behavior which apply to all functions. To start, we will examine the concepts of **increasing**, **decreasing**, and **constant**. Before defining the concepts algebraically, it is instructive to first look at them graphically. Consider the graph of the function $f$ given on the next page.

Reading from left to right, the graph 'starts' at the point $(-4, -3)$ and 'ends' at the point $(6, 5.5)$. If we imagine walking from left to right on the graph, between $(-4, -3)$ and $(-2, 4.5)$, we are walking 'uphill'; then between $(-2, 4.5)$ and $(3, -8)$, we are walking 'downhill'; and between $(3, -8)$ and $(4, -6)$, we are walking 'uphill' once more. From $(4, -6)$ to $(5, -6)$, we 'level off', and then resume walking 'uphill' from $(5, -6)$ to $(6, 5.5)$. In other words, for the $x$ values between $-4$ and $-2$ (inclusive), the $y$-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y = f(x)$, the $y$ values on the graph are the function values, and we say that the function $f$ is **increasing** on the interval $[-4, -2]$. Analogously, we say that $f$ is **decreasing** on the interval $[-2, 3]$ increasing once more on the interval $[3, 4]$, **constant** on $[4, 5]$, and finally increasing once again on $[5, 6]$. It is extremely important to notice that the behavior (increasing, decreasing or constant) occurs on an interval on the $x$-axis. When we say that the function $f$ is increasing on $[-4, -2]$ we do not mention the actual $y$ values that $f$ attains along the way. Thus, we report **where** the behavior occurs, not to what extent the behavior occurs.[6] Also notice that we do not say that a function is increasing, decreasing or constant at a single $x$ value. In fact,

[5] Or, in other words, don't rely too heavily on the machine!

[6] The notions of how quickly or how slowly a function increases or decreases are explored in Calculus.

we would run into serious trouble in our previous example if we tried to do so because $x = -2$ is contained in an interval on which $f$ was increasing and one on which it is decreasing. (There's more on this issue and many others in the exercises.)




The graph of $y = f(x)$

We're now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

DEFINITION 2.5. Suppose $f$ is a function defined on an interval $I$. We say $f$ is:

- **increasing** on $I$ if and only if $f(a) < f(b)$ for all real numbers $a$, $b$ in $I$ with $a < b$.
- **decreasing** on $I$ if and only if $f(a) > f(b)$ for all real numbers $a$, $b$ in $I$ with $a < b$.
- **constant** on $I$ if and only if $f(a) = f(b)$ for all real numbers $a$, $b$ in $I$.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing, and constant as stated in Definition 2.5 agree with our graphical descriptions given earlier. You should look back through the examples and exercise sets in previous sections where graphs were given to

see if you can determine the intervals on which the functions are increasing, decreasing or constant. Can you find an example of a function for which none of the concepts in Definition 2.5 apply?

Now let's turn our attention to a few of the points on the graph. Clearly the point $(-2, 4.5)$ does not have the largest $y$ value of all of the points on the graph of $f$ – indeed that honor goes to $(6, 5.5)$ – but $(-2, 4.5)$ should get some sort of consolation prize for being 'the top of the hill' between $x = -4$ and $x = 3$. We say that the function $f$ has a **local maximum**[7] at the point $(-2, 4.5)$, because the $y$-coordinate 4.5 is the largest $y$-value (hence, function value) on the curve 'near'[8] $x = -2$. Similarly, we say that the function $f$ has a **local minimum**[9] at the point $(3, -8)$, since the $y$-coordinate $-8$ is the smallest function value near $x = 3$. Although it is tempting to say that local extrema[10] occur when the function changes from increasing to decreasing or vice versa, it is not a precise enough way to define the concepts for the needs of Calculus. At the risk of being pedantic, we will present the traditional definitions and thoroughly vet the pathologies they induce in the exercises. We have one last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph, we see the largest $y$ value (hence the largest function value) is 5.5 at $x = 6$. In this case, we say the **maximum**[11] of $f$ is 5.5; similarly, the **minimum**[12] of $f$ is $-8$. We formalize these concepts in the following definitions.

Definition 2.6. Suppose $f$ is a function with $f(a) = b$.

- We say $f$ has a **local maximum** at the point $(a, b)$ if and only if there is an open interval $I$ containing $a$ for which $f(a) \geq f(x)$ for all $x$ in $I$ different than $a$. The value $f(a) = b$ is called 'a local maximum value of $f$' in this case.
- We say $f$ has a **local minimum** at the point $(a, b)$ if and only if there is an open interval $I$ containing $a$ for which $f(a) \leq f(x)$ for all $x$ in $I$ different than $a$. The value $f(a) = b$ is called 'a local minimum value of $f$' in this case.
- The value $b$ is called the **maximum** of $f$ if $b \geq f(x)$ for all $x$ in the domain of $f$.
- The value $b$ is called the **minimum** of $f$ if $b \leq f(x)$ for all $x$ in the domain of $f$.

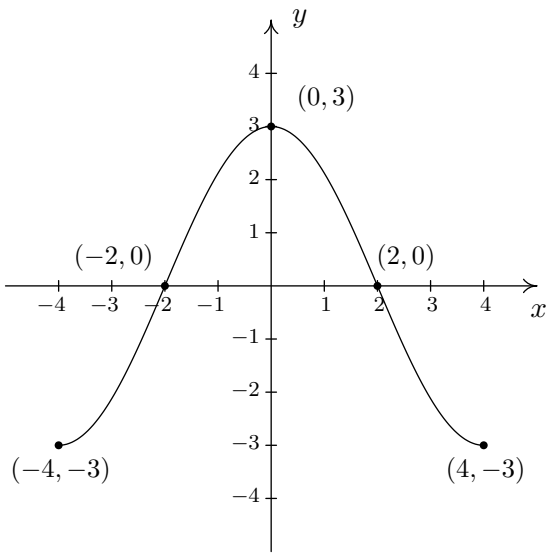
It's important to note that not every function will have all of these features. Indeed, it is possible to have a function with no local or absolute extrema at all! (Any ideas of what such a

[7] Also called 'relative maximum'.
[8] We will make this more precise in a moment.
[9] Also called a 'relative minimum'.
[10] 'Maxima' is the plural of 'maximum' and 'mimima' is the plural of 'minimum'. 'Extrema' is the plural of 'extremum' which combines maximum and minimum.
[11] Sometimes called the 'absolute' or 'global' maximum.
[12] Again, 'absolute' or 'global' minimum can be used.

function's graph would have to look like?) We shall see in the exercises examples of functions which have one or two, but not all, of these features, some that have instances of each type of extremum and some functions that seem to defy common sense. In all cases, though, we shall adhere to the algebraic definitions above as we explore the wonderful diversity of graphs that functions provide to us.

Here is the 'tame' example which was promised earlier. It summarizes all of the concepts presented in this section as well as some from previous sections so you should spend some time thinking deeply about it before proceeding to the exercises.

EXAMPLE 2.4.4. Given the graph of $y = f(x)$ below, answer all of the following questions.




1. Find the domain of $f$.
2. Find the range of $f$.
3. Determine $f(2)$.
4. List the $x$-intercepts, if any exist.
5. List the $y$-intercepts, if any exist.
6. Find the zeros of $f$.
7. Solve $f(x) < 0$.
8. Determine the number of solutions to the equation $f(x) = 1$.
9. List the intervals on which $f$ is increasing.
10. List the intervals on which $f$ is decreasing.
11. List the local maximums, if any exist.
12. List the local minimums, if any exist.
13. Find the maximum, if it exists.
14. Find the minimum, if it exists.
15. Does $f$ appear to be even, odd, or neither?

SOLUTION.

1. To find the domain of $f$, we proceed as in Section 2.1. By projecting the graph to the $x$-axis, we see the portion of the $x$-axis which corresponds to a point on the graph is everything from $-4$ to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the $y$-axis. We see that the $y$ values from $-3$ to 3, inclusive, constitute the range of $f$. Hence, our answer is $[-3, 3]$.

3. Since the graph of $f$ is the graph of the equation $y = f(x)$, $f(2)$ is the $y$-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

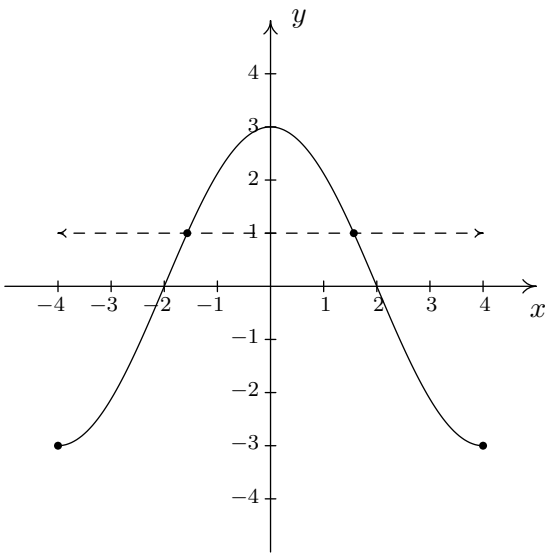
4. The $x$-intercepts are the points on the graph with $y$-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

5. The $y$-intercept is the point on the graph with $x$-coordinate 0, namely $(0, 3)$.

6. The zeros of $f$ are the $x$-coordinates of the $x$-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

7. To solve $f(x) < 0$, we look for the $x$ values of the points on the graph where the $y$-coordinate is less than 0. Graphically, we are looking where the graph is **below** the $x$-axis. This happens for the $x$ values from $-4$ to $-2$ and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

8. To find where $f(x) = 1$, we look for points on the graph where the $y$-coordinate is 1. Even though these points aren't specified, we see that the curve has two points with a $y$ value of 1, as seen in the graph below. That means there are two solutions to $f(x) = 1$.




9. As we move from left to right, the graph rises from $(-4, -3)$ to $(0, 3)$. This means $f$ is increasing on the interval $[-4, 0]$. (Remember, the answer here is an interval on the $x$-axis.)

10. As we move from left to right, the graph falls from $(0, 3)$ to $(4, -3)$. This means $f$ is decreasing on the interval $[0, 4]$. (Remember, the answer here is an interval on the $x$-axis.)

11. The function has its only local maximum at $(0, 3)$.

12. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval $I$ which contains $x = -4$ such that $f(-4) < f(x)$ for all $x$ in $I$ different from $-4$. But if we put an open interval around $x = -4$ a portion of that interval will lie outside of the domain of $f$. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that $f$ has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason – no open interval around $x = 4$ stays within the domain of $f$.

13. The maximum value of $f$ is the largest $y$-coordinate which is 3.

14. The minimum value of $f$ is the smallest $y$-coordinate which is $-3$.

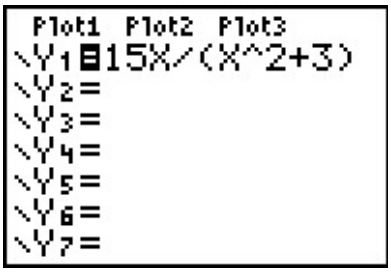
15. The graph appears to be symmetric about the $y$-axis. This suggests[13] that $f$ is even.

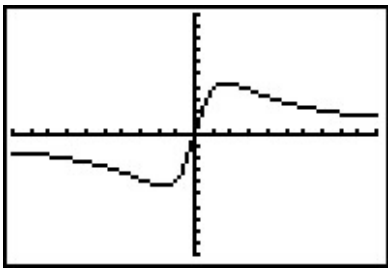
□.

With few exceptions, we will not develop techniques in College Algebra which allow us to determine the intervals on which a function is increasing, decreasing or constant or to find the local maximums and local minimums analytically; this is the business of Calculus.[14] When we have need to find such beasts, we will resort to the calculator. Most graphing calculators have 'Minimum' and 'Maximum' features which can be used to approximate these values, as demonstrated below.

EXAMPLE 2.4.5. Let $f(x) = \dfrac{15x}{x^2 + 3}$. Use a graphing calculator to approximate the intervals on which $f$ is increasing and those on which it is decreasing. Approximate all extrema.

SOLUTION. Entering this function into the calculator gives

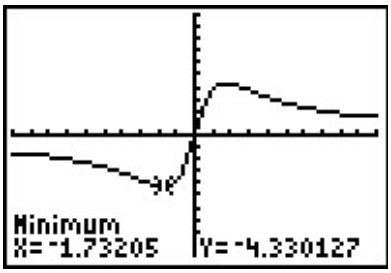


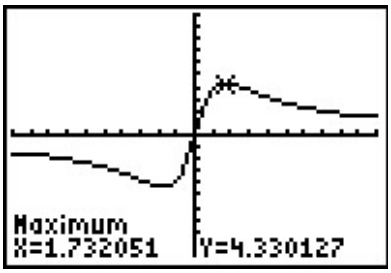



Using the Minimum and Maximum features, we get

[13] but does not prove

[14] Although, truth be told, there is only one step of Calculus involved, followed by several pages of algebra.



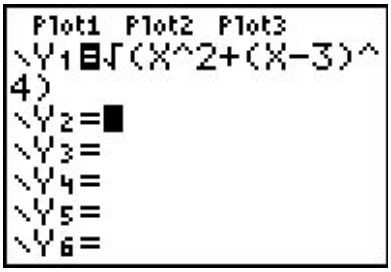

To two decimal places, $f$ appears to have its only local minimum at $(-1.73, -4.33)$ and its only local maximum at $(1, 73, 4.33)$. Given the symmetry about the origin suggested by the graph, the relation between these points shouldn't be too surprising. The function appears to be increasing on $[-1.73, 1.73]$ and decreasing on $(-\infty, -1.73] \cup [1.73, \infty)$. This makes $-4.33$ the (absolute) minimum and 4.33 the (absolute) maximum. □

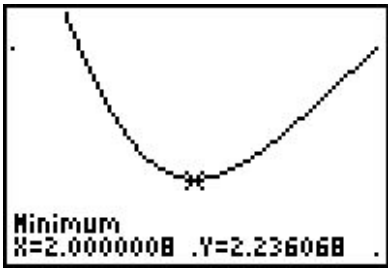
Example 2.4.6. Find the points on the graph of $y = (x-3)^2$ which are closest to the origin. Round your answers to two decimal places.

Solution. Suppose a point $(x, y)$ is on the graph of $y = (x-3)^2$. Its distance to the origin, $(0, 0)$, is given by

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2} \\ &= \sqrt{x^2 + [(x-3)^2]^2} \qquad \text{Since } y = (x-3)^2 \\ &= \sqrt{x^2 + (x-3)^4} \end{aligned}$$

Given a value for $x$, the formula $d = \sqrt{x^2 + (x-3)^4}$ is the distance from $(0, 0)$ to the point $(x, y)$ on the curve $y = (x-3)^2$. What we have defined, then, is a function $d(x)$ which we wish to minimize over all values of $x$. To accomplish this task analytically would require Calculus so as we've mentioned before, we can use a graphing calculator to find an approximate solution. Using the calculator, we enter the function $d(x)$ as shown below and graph.



Using the Minimum feature, we see above on the right that the (absolute) minimum occurs near $x = 2$. Rounding to two decimal places, we get that the minimum distance occurs when $x = 2.00$. To find the $y$ value on the parabola associated with $x = 2.00$, we substitute 2.00 into the equation to get $y = (x-3)^2 = (2.00-3)^2 = 1.00$. So, our final answer is $(2.00, 1.00)$.[15] (What does the $y$ value listed on the calculator screen mean in this problem?) □

### 2.4.2 EXERCISES

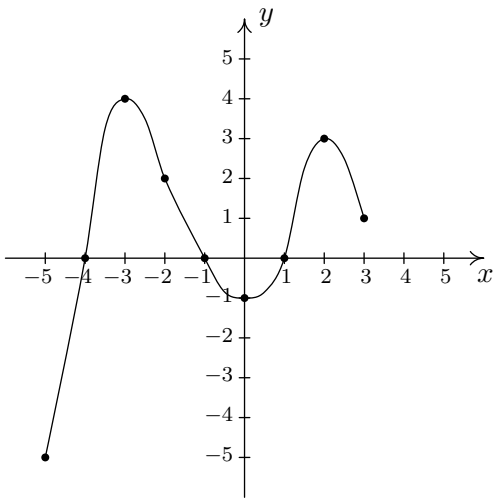
1. Sketch the graphs of the following functions. State the domain of the function, identify any intercepts and test for symmetry.

   (a) $f(x) = \dfrac{x-2}{3}$ (b) $f(x) = \sqrt{5-x}$ (c) $f(x) = \sqrt[3]{x}$ (d) $f(x) = \dfrac{1}{x^2+1}$

2. Analytically determine if the following functions are even, odd or neither.

   (a) $f(x) = 7x$
   (b) $f(x) = 7x + 2$
   (c) $f(x) = \dfrac{1}{x^3}$
   (d) $f(x) = 4$
   (e) $f(x) = 0$
   (f) $f(x) = x^6 - x^4 + x^2 + 9$
   (g) $f(x) = -x^5 - x^3 + x$
   (h) $f(x) = x^4+x^3+x^2+x+1$
   (i) $f(x) = \sqrt{5-x}$
   (j) $f(x) = x^2 - x - 6$

3. Given the graph of $y = f(x)$ below, answer all of the following questions.




[15] It seems silly to list a final answer as $(2.00, 1.00)$. Indeed, Calculus confirms that the **exact** answer to this problem is, in fact, $(2, 1)$. As you are well aware by now, the author is a pedant, and as such, uses the decimal places to remind the reader that **any** result garnered from a calculator in this fashion is an approximation, and should be treated as such.

(a) Find the domain of $f$.
(b) Find the range of $f$.
(c) Determine $f(-2)$.
(d) List the $x$-intercepts, if any exist.
(e) List the $y$-intercepts, if any exist.
(f) Find the zeros of $f$.
(g) Solve $f(x) \geq 0$.
(h) Determine the number of solutions to the equation $f(x) = 2$.
(i) List the intervals where $f$ is increasing.
(j) List the intervals where $f$ is decreasing.
(k) List the local maximums, if any exist.
(l) List the local minimums, if any exist.
(m) Find the maximum, if it exists.
(n) Find the minimum, if it exists.
(o) Is $f$ even, odd, or neither?

4. Use your graphing calculator to approximate the local and absolute extrema of the following functions. Approximate the intervals on which the function is increasing and those on which it is decreasing. Round your answers to two decimal places.

(a) $f(x) = x^4 - 3x^3 - 24x^2 + 28x + 48$
(b) $f(x) = x^{2/3}(x - 4)$
(c) $f(x) = \sqrt{9 - x^2}$
(d) $f(x) = x\sqrt{9 - x^2}$

5. Sketch the graphs of the following piecewise-defined functions.

(a) $f(x) = \begin{cases} -2x - 4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} \sqrt{x + 4} & \text{if} \quad -4 \leq x < 5 \\ \sqrt{x - 1} & \text{if} \quad x \geq 5 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -2 \\ 3 - x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \geq 2 \end{cases}$

(d) $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

6. Let $f(x) = \lfloor x \rfloor$, the greatest integer function defined in Exercise 12 in Section 2.2.

(a) Graph $y = f(x)$. Be careful to correctly describe the behavior of the graph near the integers.
(b) Is $f$ even, odd, or neither? Explain.
(c) Discuss with your classmates which points on the graph are local minimums, local maximums or both. Is $f$ ever increasing? Decreasing? Constant?

7. Use your graphing calculator to show that the following functions do not have any extrema, neither local nor absolute.

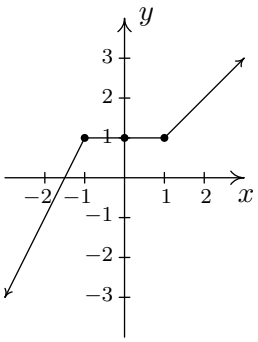
(a) $f(x) = x^3 + x - 12$ (b) $f(x) = -5x + 2$

8. In Exercise 11 in Section 2.2, we saw that the population of Sasquatch in Portage County could be modeled by the function $P(t) = \dfrac{150t}{t+15}$, where $t = 0$ represents the year 1803. Use your graphing calculator to analyze the general function behavior of $P$. Will there ever be a time when 200 Sasquatch roam Portage County?

9. One of the most important aspects of the Cartesian Coordinate Plane is its ability to put Algebra into geometric terms and Geometry into algebraic terms. We've spent most of this chapter looking at this very phenomenon and now you should spend some time with your classmates reviewing what we've done. What major results do we have that tie Algebra and Geometry together? What concepts from Geometry have we not yet described algebraically? What topics from Intermediate Algebra have we not yet discussed geometrically?

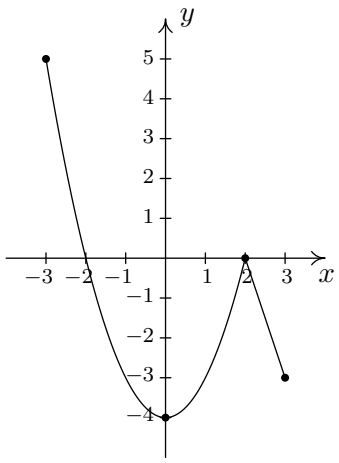
10. It's now time to "thoroughly vet the pathologies induced" by the precise definitions of local maximum and local minimum. We'll do this by providing you and your classmates a series of exercises to discuss. You will need to refer back to Definition 2.5 (Increasing, Decreasing and Constant) and Definition 2.6 (Maximum and Minimum) during the discussion.

    (a) Consider the graph of the function $f$ given below.

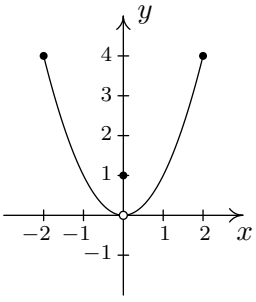



        i. Show that $f$ has a local maximum but not a local minimum at the point $(-1, 1)$.
        ii. Show that $f$ has a local minimum but not a local maximum at the point $(1, 1)$.
        iii. Show that $f$ has a local maximum AND a local minimum at the point $(0, 1)$.
        iv. Show that $f$ is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

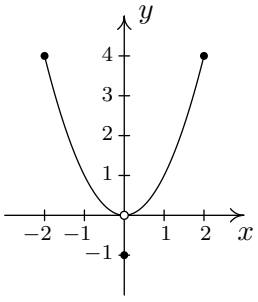
    (b) Using Example 2.4.4 as a guide, show that the function $g$ whose graph is given below does not have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute. What's unique about the point $(0, -4)$ on this graph? Also find the intervals on which $g$ is increasing and those on which $g$ is decreasing.



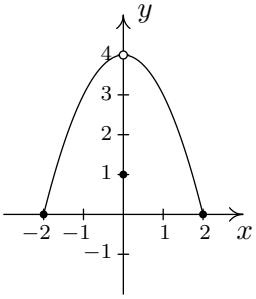

(c) We said earlier in the section that it is not good enough to say local extrema exist where a function changes from increasing to decreasing or vice versa. As a previous exercise showed, we could have local extrema when a function is constant so now we need to examine some functions whose graphs do indeed change direction. Consider the functions graphed below. Notice that all four of them change direction at an open circle on the graph. Examine each for local extrema. What is the effect of placing the "dot" on the $y$-axis above or below the open circle? What could you say if no function value was assigned to $x = 0$?



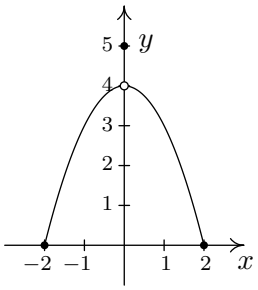

i. Function I




iii. Function III




ii. Function II

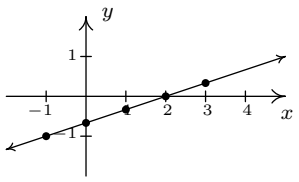



iv. Function IV

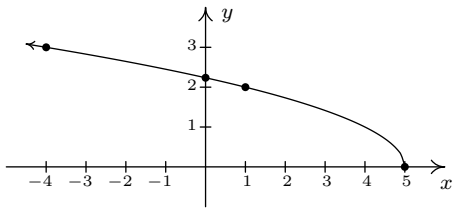
### 2.4.3 Answers

1. (a)

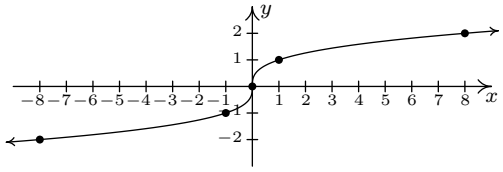
$f(x) = \dfrac{x-2}{3}$
Domain: $(-\infty, \infty)$
$x$-intercept: $(2, 0)$
$y$-intercept: $\left(0, -\frac{2}{3}\right)$
No symmetry



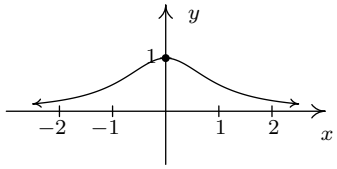

(b) $f(x) = \sqrt{5-x}$
Domain: $(-\infty, 5]$
$x$-intercept: $(5, 0)$
$y$-intercept: $(0, \sqrt{5})$
No symmetry




(c) $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
$x$-intercept: $(0, 0)$
$y$-intercept: $(0, 0)$
Symmetry about the origin




(d) $f(x) = \dfrac{1}{x^2+1}$
Domain: $(-\infty, \infty)$
No $x$-intercepts
$y$-intercept: $(0, 1)$
Symmetry about the $y$-axis




2. (a) $f(x) = 7x$ is odd
(b) $f(x) = 7x + 2$ is neither
(c) $f(x) = \dfrac{1}{x^3}$ is odd
(d) $f(x) = 4$ is even
(e) $f(x) = 0$ is even **and** odd
(f) $f(x) = x^6 - x^4 + x^2 + 9$ is even
(g) $f(x) = -x^5 - x^3 + x$ is odd
(h) $f(x) = x^4 + x^3 + x^2 + x + 1$ is neither
(i) $f(x) = \sqrt{5-x}$ is neither
(j) $f(x) = x^2 - x - 6$ is neither

3. (a) $[-5, 3]$
(b) $[-5, 4]$
(c) $f(-2) = 2$
(d) $(-4, 0)$, $(-1, 0)$, $(1, 0)$
(e) $(0, -1)$
(f) $-4$, $-1$, $1$
(g) $[-4, -1]$, $[1, 3]$
(h) 4
(i) $[-5, -3]$, $[0, 2]$
(j) $[-3, 0]$, $[2, 3]$
(k) $(-3, 4)$, $(2, 3)$
(l) $(0, -1)$
(m) 4
(n) $-5$
(o) Neither

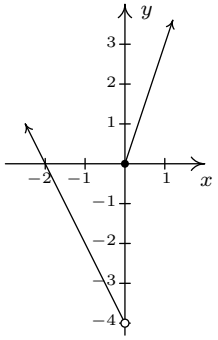
4. (a) No absolute maximum
Absolute minimum $f(4.55) \approx -175.46$
Local minimum at $(-2.84, -91.32)$
Local maximum at $(0.54, 55.73)$
Local minimum at $(4.55, -175.46)$
Increasing on $[-2.84, 0.54], [4.55, \infty)$
Decreasing on $(-\infty, -2.84], [0.54, 4.55]$

(b) No absolute maximum
No absolute minimum
Local maximum at $(0, 0)$
Local minimum at $(1.60, -3.28)$
Increasing on $(-\infty, 0], [1.60, \infty)$
Decreasing on $[0, 1.60]$

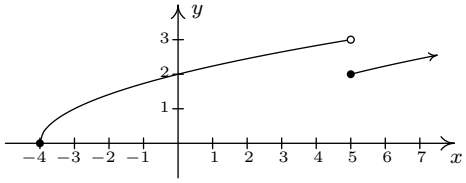
(c) Absolute maximum $f(0) = 3$
Absolute minimum $f(\pm 3) = 0$
Local maximum at $(0, 3)$
No local minimum
Increasing on $[-3, 0]$
Decreasing on $[0, 3]$

(d) Absolute maximum $f(2.12) \approx 4.50$
Absolute minimum $f(-2.12) \approx -4.50$
Local maximum $(2.12, 4.50)$
Local minimum $(-2.12, -4.50)$
Increasing on $[-2.12, 2.12]$
Decreasing on $[-3, -2.12], [2.12, 3]$

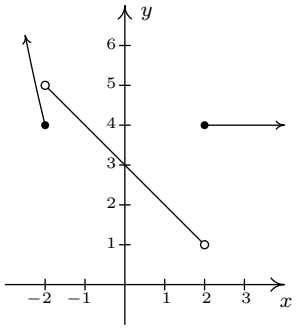
5. (a)



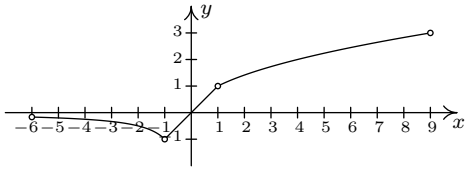

(b)



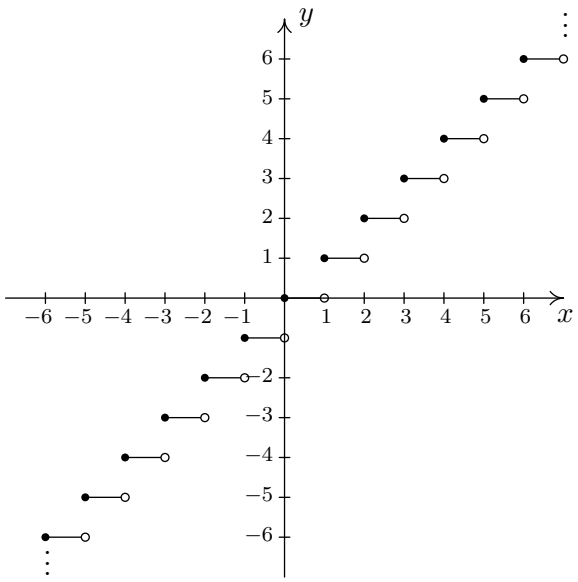

(c)




(d)




6. (a)

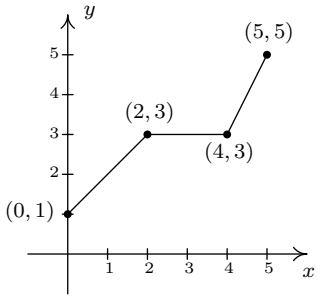



The graph of $f(x) = \lfloor x \rfloor$.

(b) Note that $f(1.1) = 1$, but $f(-1.1) = -2$, and so $f$ is neither even nor odd.

## 2.5 TRANSFORMATIONS

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections, and scalings, and we will present them in that order. Suppose the graph below is the complete graph of $f$.



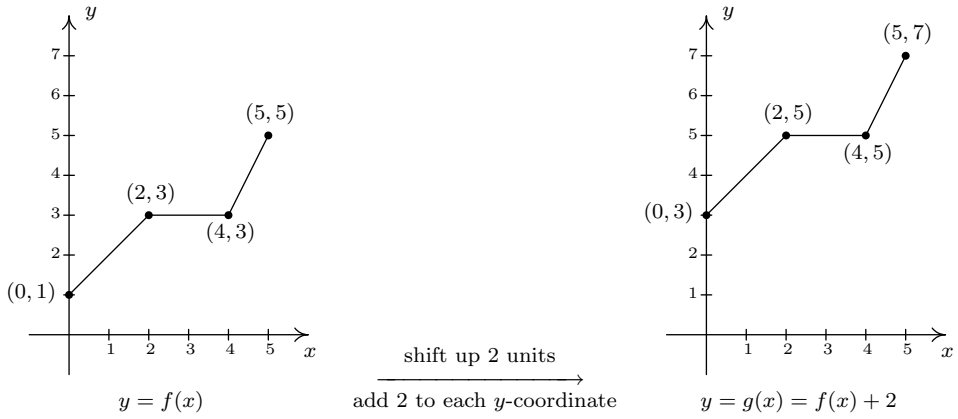

$y = f(x)$

The Fundamental Graphing Principle for Functions says that for a point $(a, b)$ to be on the graph, $f(a) = b$. In particular, we know $f(0) = 1$, $f(2) = 3$, $f(4) = 3$ and $f(5) = 5$. Suppose we wanted to graph the function defined by the formula $g(x) = f(x) + 2$. Let's take a minute to remind ourselves of what $g$ is doing. We start with an input $x$ to the function $f$ and we obtain the output $f(x)$. The function $g$ takes the output $f(x)$ and adds 2 to it. In order to graph $g$, we need

to graph the points $(x, g(x))$. How are we to find the values for $g(x)$ without a formula for $f(x)$? The answer is that we don't need a *formula* for $f(x)$, we just need the *values* of $f(x)$. The values of $f(x)$ are the $y$ values on the graph of $y = f(x)$. For example, using the points indicated on the graph of $f$, we can make the following table.

| $x$ | $(x, f(x))$ | $f(x)$ | $g(x) = f(x) + 2$ | $(x, g(x))$ |
|---|---|---|---|---|
| 0 | $(0, 1)$ | 1 | 3 | $(0, 3)$ |
| 2 | $(2, 3)$ | 3 | 5 | $(2, 5)$ |
| 4 | $(4, 3)$ | 3 | 5 | $(4, 5)$ |
| 5 | $(5, 5)$ | 5 | 7 | $(5, 7)$ |

In general, if $(a, b)$ is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b + 2)$ is on the graph of $g$. In other words, to obtain the graph of $g$, we add 2 to the $y$-coordinate of each point on the graph of $f$. Geometrically, adding 2 to the $y$-coordinate of a point moves the point 2 units **above** its previous location. Adding 2 to every $y$-coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before. We have the results side-by-side below.




$y = f(x)$ $\qquad$ $y = g(x) = f(x) + 2$

You'll note that the domain of $f$ and the domain of $g$ are the same, namely $[0, 5]$, but that the range of $f$ is $[1, 5]$ while the range of $g$ is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the $y$-coordinates on the graph of $f$, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of $j$ is the same as $f$, but the range of $j$ is $[-1, 3]$. What we have discussed is generalized in the following theorem.

THEOREM 2.2. **Vertical Shifts.** Suppose $f$ is a function and $k$ is a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ **up** $k$ units by **adding** $k$ to the ***y*-coordinates** of the points on the graph of $f$.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ **down** $k$ units by **subtracting** $k$ from the ***y*-coordinates** of the points on the graph of $f$.

The key to understanding Theorem 2.2 and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If $(a, b)$ is on the graph of $f$, then $f(a) = b$. Substituting $x = a$ into the equation $y = f(x) + k$ gives $y = f(a) + k = b + k$. Hence, $(a, b + k)$ is on the graph of $y = f(x) + k$, and we have the result. In the language of 'inputs' and 'outputs', Theorem 2.2 can be paraphrased as "Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively". So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function.[1] Let's try to generate a table of values of $g$ based on those we know for $f$. We quickly find that we run into some difficulties.

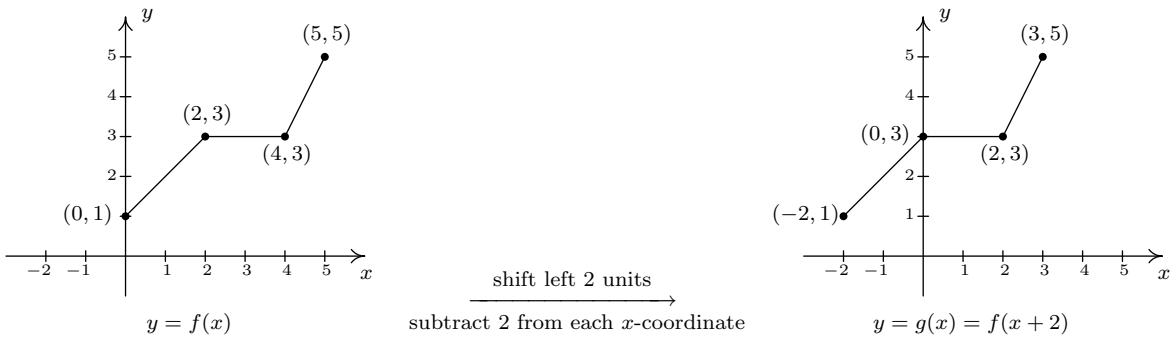
| $x$ | $(x, f(x))$ | $f(x)$ | $g(x) = f(x + 2)$ | $(x, g(x))$ |
|---|---|---|---|---|
| 0 | $(0, 1)$ | 1 | $f(0 + 2) = f(2) = 3$ | $(0, 3)$ |
| 2 | $(2, 3)$ | 3 | $f(2 + 2) = f(4) = 3$ | $(2, 3)$ |
| 4 | $(4, 3)$ | 3 | $f(4 + 2) = f(6) =?$ | |
| 5 | $(5, 5)$ | 5 | $f(5 + 2) = f(7) =?$ | |

When we substitute $x = 4$ into the formula $g(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$ which doesn't exist because the domain of $f$ is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$. What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of $g$, we need to figure out what value of $x$ we must substitute into $g(x) = f(x + 2)$ so that the quantity $x + 2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ on the graph of $g$. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0 + 2) = f(2) = 3$. Continuing in this fashion, we get

[1] We have spent a lot of time in this text showing you that $f(x + 2)$ and $f(x) + 2$ are, in general, wildly different algebraic animals. We will see momentarily that the geometry is also dramatically different.

| $x$ | $x+2$ | $g(x) = f(x+2)$ | $(x, g(x))$ |
|---|---|---|---|
| $-2$ | $0$ | $g(-2) = f(0) = 1$ | $(-2, 1)$ |
| $0$ | $2$ | $g(0) = f(2) = 3$ | $(0, 3)$ |
| $2$ | $4$ | $g(2) = f(4) = 3$ | $(2, 3)$ |
| $3$ | $5$ | $g(3) = f(5) = 5$ | $(3, 5)$ |

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if $(a, b)$ is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a - 2) = f((a - 2) + 2) = f(a) = b$. As such, $(a - 2, b)$ is on the graph of $y = g(x)$. The point $(a - 2, b)$ is exactly 2 units to the **left** of the point $(a, b)$ so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.




Note that while the ranges of $f$ and $g$ are the same, the domain of $g$ is $[-2, 3]$ whereas the domain of $f$ is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x - 2)$, we would find ourselves *adding* 2 to all of the $x$ values of the points on the graph of $y = f(x)$ to effect a shift to the **right** 2 units. Generalizing, we have the following result.

THEOREM 2.3. **Horizontal Shifts.** Suppose $f$ is a function and $h$ is a positive number.

- To graph $y = f(x + h)$, shift the graph of $y = f(x)$ **left** $h$ units by **subtracting** $h$ from the ***x*-coordinates** of the points on the graph of $f$.
- To graph $y = f(x - h)$, shift the graph of $y = f(x)$ **right** $h$ units by **adding** $h$ to the ***x*-coordinates** of the points on the graph of $f$.

In other words, Theorem 2.3 says adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems 2.2 and 2.3 present a theme which will run common throughout the section: changes to the **outputs** from a function affect the ***y*-coordinates** of the graph, resulting in some kind of vertical change; changes to the **inputs** to a function affect the ***x*-coordinates** of the graph, resulting in some kind of horizontal change.

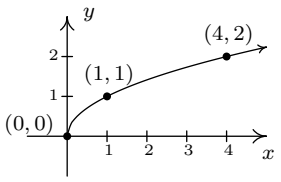
EXAMPLE 2.5.1.

1. Graph $f(x) = \sqrt{x}$. Plot at least three points.
2. Use your graph in 1 to graph $g(x) = \sqrt{x} - 1$.
3. Use your graph in 1 to graph $j(x) = \sqrt{x-1}$.
4. Use your graph in 1 to graph $m(x) = \sqrt{x+3} - 2$.

SOLUTION.

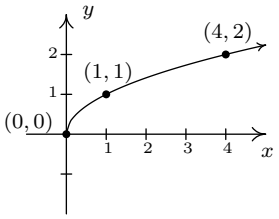
1. Owing to the square root, the domain of $f$ is $x \geq 0$, or $[0, \infty)$. We choose perfect squares to build our table and graph below. From the graph we verify the domain of $f$ is $[0, \infty)$ and the range of $f$ is also $[0, \infty)$.

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| 0 | 0 | $(0, 0)$ |
| 1 | 1 | $(1, 1)$ |
| 4 | 2 | $(4, 2)$ |



$y = f(x) = \sqrt{x}$

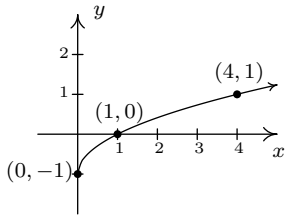
2. The domain of $g$ is the same as the domain of $f$, since the only condition on both functions is that $x \geq 0$. If we compare the formula for $g(x)$ with $f(x)$, we see that $g(x) = f(x) - 1$. In other words, we have subtracted 1 from the output of the function $f$. By Theorem 2.2, we know that in order to graph $g$, we shift the graph of $f$ **down** one unit by **subtracting** 1 from each of the $y$-coordinates of the points on the graph of $f$. Applying this to the three points we have specified on the graph, we move $(0, 0)$ to $(0, -1)$, $(1, 1)$ to $(1, 0)$, and $(4, 2)$ to $(4, 1)$. The rest of the points follow suit, and we connect them with the same basic shape as before. We confirm the domain of $g$ is $[0, \infty)$ and find the range of $g$ to be $[-1, \infty)$.



$y = f(x) = \sqrt{x}$

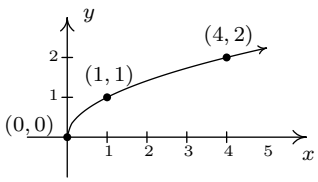
shift down 1 unit

subtract 1 from each $y$-coordinate



$y = g(x) = \sqrt{x} - 1$

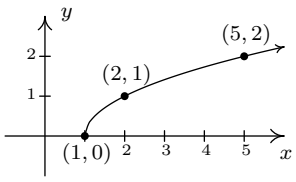
3. Solving $x - 1 \geq 0$ gives $x \geq 1$, so the domain of $j$ is $[1, \infty)$. To graph $j$, we note that $j(x) = f(x - 1)$. In other words, we are subtracting 1 from the *input* of $f$. According to Theorem 2.3, this induces a shift to the **right** of the graph of $f$. We **add** 1 to the $x$-coordinates of the points on the graph of $f$ and get the result below. The graph reaffirms the domain of $j$ is $[1, \infty)$ and tells us that the range is $[0, \infty)$.




$y = f(x) = \sqrt{x}$

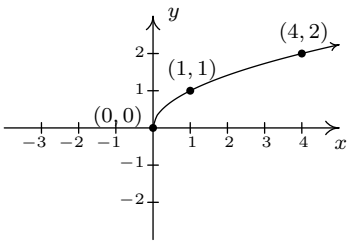
shift right 1 unit

add 1 to each $x$-coordinate




$y = j(x) = \sqrt{x - 1}$

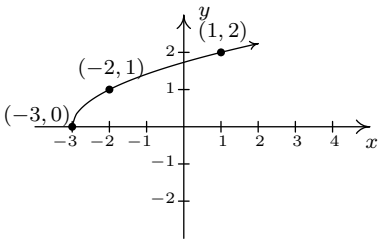
4. To find the domain of $m$, we solve $x + 3 \geq 0$ and get $[-3, \infty)$. Comparing the formulas of $f(x)$ and $m(x)$, we have $m(x) = f(x + 3) - 2$. We have 3 being added to an input, indicating a horizontal shift, and 2 being subtracted from an output, indicating a vertical shift. We leave it to the reader to verify that, in this particular case, the order in which we perform these transformations is immaterial; we will arrive at the same graph regardless as to which transformation we apply first.[2] We follow the convention 'inputs first',[3] and to that end we first tackle the horizontal shift. Letting $m_1(x) = f(x + 3)$ denote this intermediate step, Theorem 2.3 tells us that the graph of $y = m_1(x)$ is the graph of $f$ shifted to the **left** 3 units. Hence, we **subtract** 3 from each of the $x$-coordinates of the points on the graph of $f$.




$y = f(x) = \sqrt{x}$

shift left 3 units

subtract 3 from each $x$-coordinate

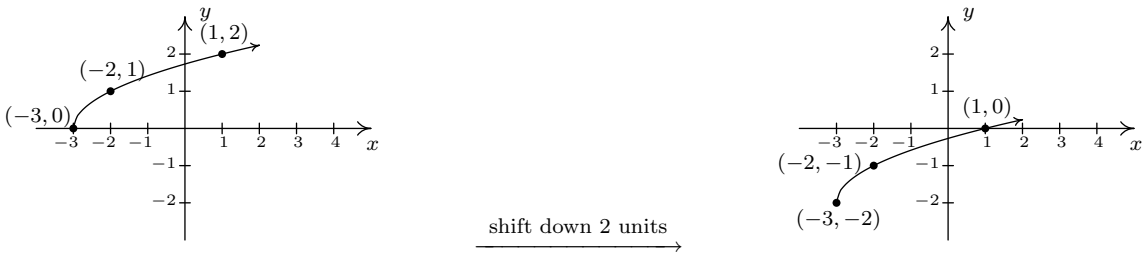



$y = m_1(x) = f(x + 3) = \sqrt{x + 3}$

Since $m(x) = f(x + 3) - 2$ and $f(x + 3) = m_1(x)$, we have $m(x) = m_1(x) - 2$. We can apply Theorem 2.2 and obtain the graph of $m$ by **subtracting** 2 from the $y$-coordinates of each of the points on the graph of $m_1(x)$. The graph verifies that the domain of $m$ is $[-3, \infty)$ and we find the range of $m$ is $[-2, \infty)$.

[2] We shall see in the next example that order is generally important when applying more than one transformation to a graph.

[3] We could equally have chosen the convention 'outputs first'.




$y = m_1(x) = f(x+3) = \sqrt{x+3}$ $\qquad$ $y = m(x) = m_1(x) - 2 = \sqrt{x+3} - 2$

Keep in mind that we can check our answer to any of these kinds of problems by showing that any of the points we've moved lie on the graph of our final answer. For example, we can check that $(-3, -2)$ is on the graph of $m$, by computing $m(-3) = \sqrt{(-3)+3} - 2 = \sqrt{0} - 2 = -2$ ✓ □

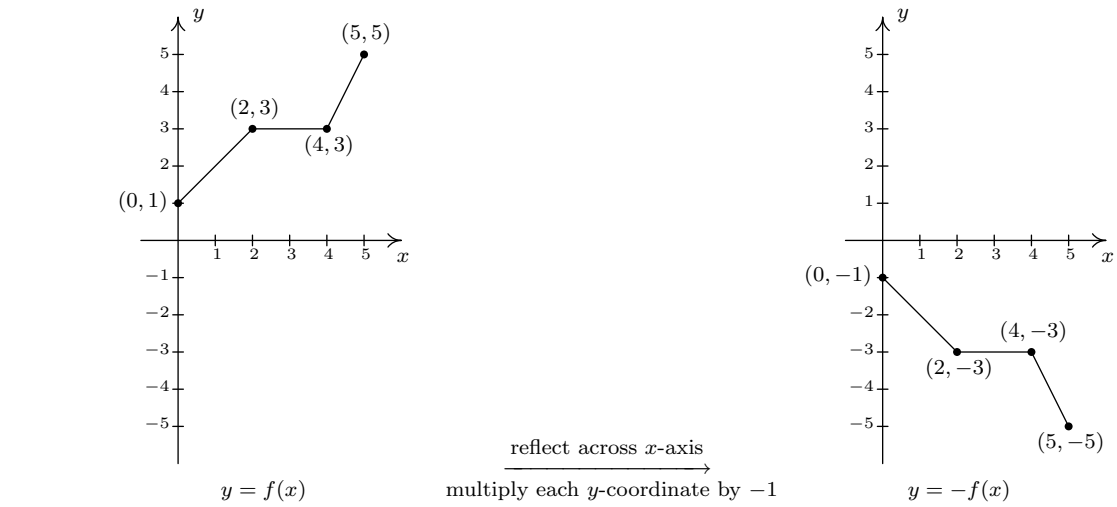
We now turn our attention to reflections. We know from Section 1.1 that to reflect a point $(x, y)$ across the $x$-axis, we replace $y$ with $-y$. If $(x, y)$ is on the graph of $f$, then $y = f(x)$, so replacing $y$ with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of $f$ reflected across the $x$-axis. Similarly, the graph of $y = f(-x)$ is the graph of $f$ reflected across the $y$-axis. Returning to inputs and outputs, multiplying the output from a function by $-1$ reflects its graph across the $x$-axis, while multiplying the input to a function by $-1$ reflects the graph across the $y$-axis.[4]

THEOREM 2.4. **Reflections.** Suppose $f$ is a function.

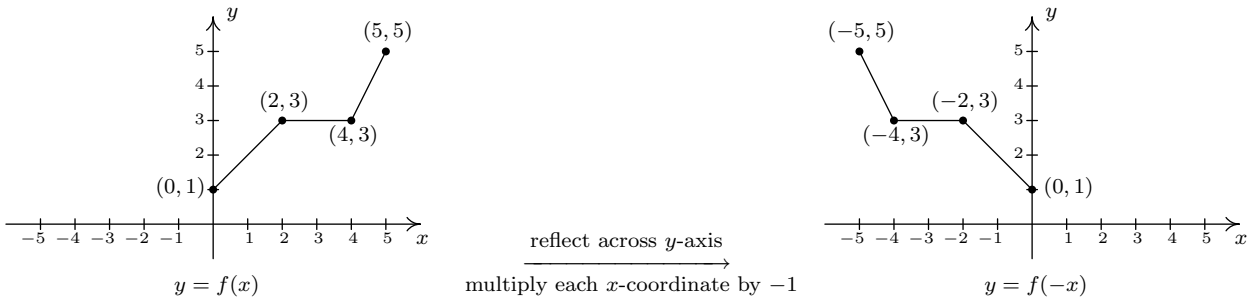
- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the ***x*-axis** by multiplying the ***y*-coordinates** of the points on the graph of $f$ by $-1$.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the ***y*-axis** by multiplying the ***x*-coordinates** of the points on the graph of $f$ by $-1$.

Applying Theroem 2.4 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of $f$ about the $x$-axis

[4]The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section 2.4 to test if a function was even, odd, or neither. The interested reader is invited to explore the role of reflections and symmetry of functions. What happens if you reflect an even function across the $y$-axis? What happens if you reflect an odd function across the $y$-axis? What about the $x$-axis?




By reflecting the graph of $f$ across the $y$-axis, we obtain the graph of $y = f(-x)$.




With the addition of reflections, it is now more important than ever to consider the **order** of transformations, as the next example illustrates.

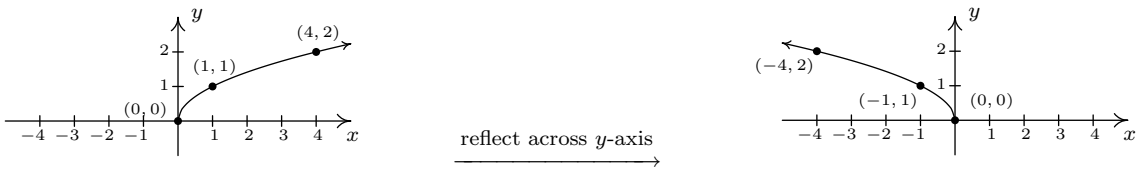
EXAMPLE 2.5.2. Let $f(x) = \sqrt{x}$. Use the graph of $f$ from Example 2.5.1 to graph the following functions below. Also, state their domains and ranges.

1. $g(x) = \sqrt{-x}$
2. $j(x) = \sqrt{3 - x}$
3. $m(x) = 3 - \sqrt{x}$

SOLUTION.

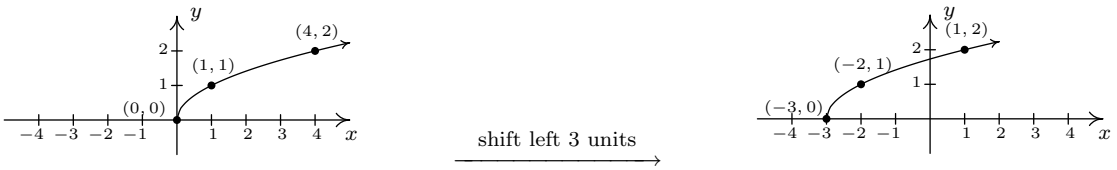
1. The mere sight of $\sqrt{-x}$ usually causes alarm, if not panic. When we discussed domains in Section 2.2, we clearly banished negatives from the radicals of even roots. However, we must remember that $x$ is a variable, and as such, the quantity $-x$ isn't always negative. For

example, if $x = -4$, $-x = 4$, thus $\sqrt{-x} = \sqrt{-(-4)} = 2$ is perfectly well-defined. To find the domain analytically, we set $-x \geq 0$ which gives $x \leq 0$, so that the domain of $g$ is $(-\infty, 0]$. Since $g(x) = f(-x)$, Theorem 2.4 tells us the graph of $g$ is the reflection of the graph of $f$ across the $y$-axis. We can accomplish this by multiplying each $x$-coordinate on the graph of $f$ by $-1$, so that the points $(0, 0)$, $(1, 1)$, and $(4, 2)$ move to $(0, 0)$, $(-1, 1)$, and $(-4, 2)$, respectively. Graphically, we see that the domain of $g$ is $(-\infty, 0]$ and the range of $g$ is the same as the range of $f$, namely $[0, \infty)$.




$y = f(x) = \sqrt{x}$ $\qquad$ $y = g(x) = f(-x) = \sqrt{-x}$

2. To determine the domain of $j(x) = \sqrt{3 - x}$, we solve $3 - x \geq 0$ and get $x \leq 3$, or $(-\infty, 3]$. To determine which transformations we need to apply to the graph of $f$ to obtain the graph of $j$, we rewrite $j(x) = \sqrt{-x + 3} = f(-x + 3)$. Comparing this formula with $f(x) = \sqrt{x}$, we see that not only are we multiplying the input $x$ by $-1$, which results in a reflection across the $y$-axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point $(4, 2)$ on the graph of $f$. We refer to the discussion leading up to Theorem 2.3. We know $f(4) = 2$ and wish to find the point on $y = j(x) = f(-x + 3)$ which corresponds to $(4, 2)$. We set $-x + 3 = 4$ and solve. Our first step is to subtract 3 from both sides to get $-x = 1$. Subtracting 3 from the $x$-coordinate 4 is shifting the point $(4, 2)$ to the left. From $-x = 1$, we then multiply[5] both sides by $-1$ to get $x = -1$. Multiplying the $x$-coordinate by $-1$ corresponds to reflecting the point about the $y$-axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the $y$-axis. Starting with $f(x) = \sqrt{x}$, we let $j_1(x)$ be the intermediate function which shifts the graph of $f$ 3 units to the left, $j_1(x) = f(x + 3)$.

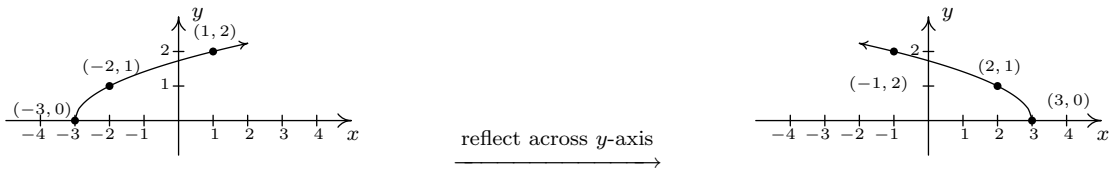



$y = f(x) = \sqrt{x}$ $\qquad$ $y = j_1(x) = f(x + 3) = \sqrt{x + 3}$

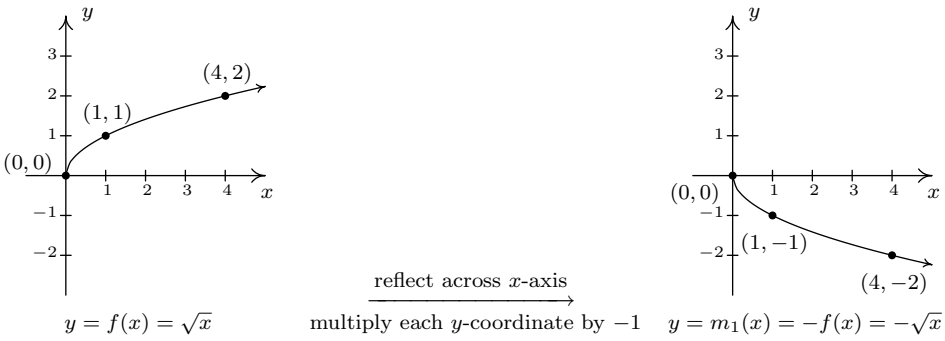
To obtain the function $j$, we reflect the graph of $j_1$ about $y$-axis. Theorem 2.4 tells us we have $j(x) = j_1(-x)$. Putting it all together, we have $j(x) = j_1(-x) = f(-x + 3) = \sqrt{-x + 3}$,

[5] Or divide - it amounts to the same thing.

which is what we want.[6] From the graph, we confirm the domain of $j$ is $(-\infty, 3]$ and we get the range is $[0, \infty)$.

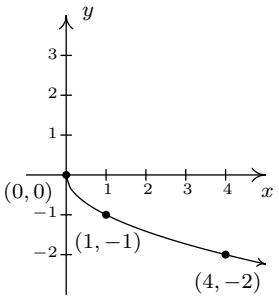



3. The domain of $m$ works out to be the domain of $f$, $[0, \infty)$. Rewriting $m(x) = -\sqrt{x} + 3$, we see $m(x) = -f(x) + 3$. Since we are multiplying the output of $f$ by $-1$ and then adding 3, we once again have two transformations to deal with: a reflection across the $x$-axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of $m$ which corresponds to $(4, 2)$ on the graph of $f$. Since in the formula for $m(x)$, the input to $f$ is just $x$, we substitute to find $m(4) = -f(4) + 3 = -2 + 3 = 1$. Hence, $(4, 1)$ is the corresponding point on the graph of $m$. If we closely examine the arithmetic, we see that we first multiply $f(4)$ by $-1$, which corresponds to the reflection across the $x$-axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function $m_1(x) = -f(x)$ to take care of the reflection, we get




To shift the graph of $m_1$ up 3 units, we set $m(x) = m_1(x) + 3$. Since $m_1(x) = -f(x)$, when we put it all together, we get $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$. We see from the graph that the range of $m$ is $(-\infty, 3]$.

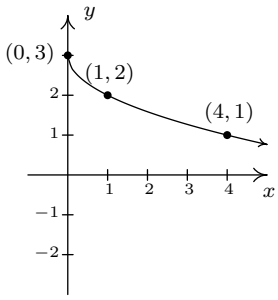
[6] If we had done the reflection first, then $j_1(x) = f(-x)$. Following this by a shift left would give us $j(x) = j_1(x + 3) = f(-(x + 3)) = f(-x - 3) = \sqrt{-x - 3}$ which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function $j(x)$. We leave it to the reader to verify the details.




$y = m_1(x) = -\sqrt{x}$

shift up 3 units

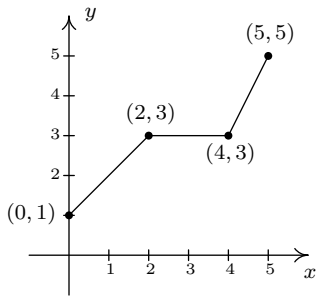
add 3 to each $y$-coordinate




$y = m(x) = m_1(x) + 3 = -\sqrt{x} + 3$

□

We now turn our attention to our last class of transformations, scalings. Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for $g$ as before.



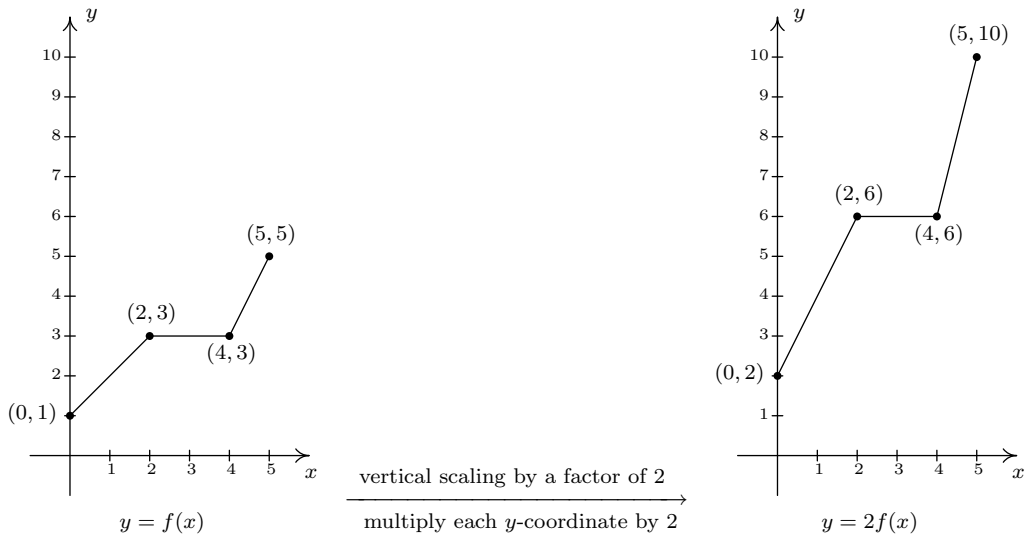

$y = f(x)$

| $x$ | $(x, f(x))$ | $f(x)$ | $g(x) = 2f(x)$ | $(x, g(x))$ |
|---|---|---|---|---|
| 0 | $(0, 1)$ | 1 | 2 | $(0, 2)$ |
| 2 | $(2, 3)$ | 3 | 6 | $(2, 6)$ |
| 4 | $(4, 3)$ | 3 | 6 | $(4, 6)$ |
| 5 | $(5, 5)$ | 5 | 10 | $(5, 10)$ |

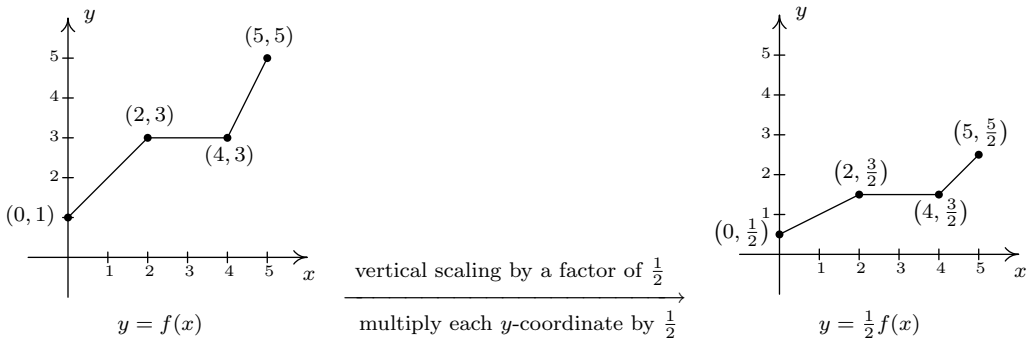
In general, if $(a, b)$ is on the graph of $f$, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of $g$. In other words, to obtain the graph of $g$, we multiply all of the $y$-coordinates of the points on the graph of $f$ by 2. Multiplying all of the $y$-coordinates of all of the points on the graph of $f$ by 2 causes what is known as a 'vertical scaling[7] by a factor of 2', and the results are

[7] Also called a 'vertical stretch', 'vertical expansion' or 'vertical dilation' by a factor of 2.

given below.




If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the $y$-coordinates of the points on the graph of $f$ by $\frac{1}{2}$. This creates a 'vertical scaling[8] by a factor of $\frac{1}{2}$' as seen below.




These results are generalized in the following theorem.

[8]Also called 'vertical shrink,''vertical compression' or 'vertical contraction' by a factor of 2.

THEOREM 2.5. **Vertical Scalings.** Suppose $f$ is a function and $a > 0$. To graph $y = af(x)$, **multiply** all of the ***y*-coordinates** of the points on the graph of $f$ by $a$. We say the graph of $f$ has been **vertically scaled** by a factor of $a$.

- If $a > 1$, we say the graph of $f$ has undergone a vertical stretch (expansion, dilation) by a factor of $a$.
- If $0 < a < 1$, we say the graph of $f$ has undergone a vertical shrink (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about Theorem 2.5 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$.' This is why we have written the descriptions 'stretch by a factor of $a$' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem. Second, in terms of inputs and outputs, Theorem 2.5 says multiplying the *outputs* from a function by positive number $a$ causes the graph to be vertically scaled by a factor of $a$. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

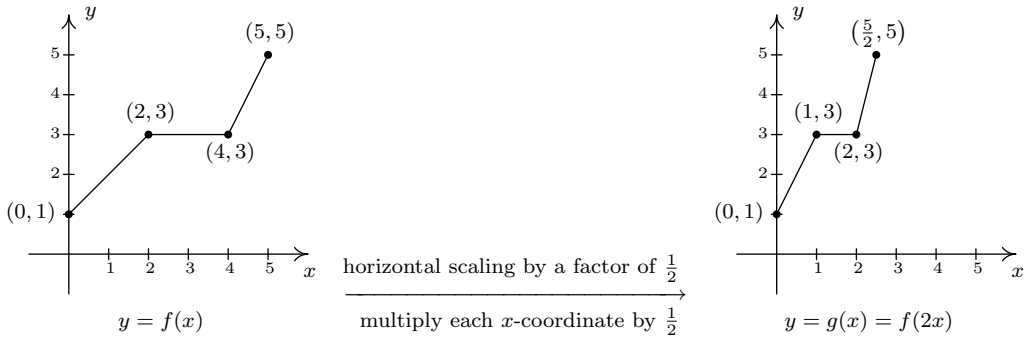
Referring to the graph of $f$ given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to $f$ by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 2.3, as seen in the table on the left below. We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on $g$ which corresponds to the point $(2, 3)$ on the graph of $f$, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of $g$. Continuing in this fashion, we obtain the table on the lower right.

| $x$ | $(x, f(x))$ | $f(x)$ | $g(x) = f(2x)$ | $(x, g(x))$ |
|---|---|---|---|---|
| 0 | $(0, 1)$ | 1 | $f(2 \cdot 0) = f(0) = 1$ | $(0, 1)$ |
| 2 | $(2, 3)$ | 3 | $f(2 \cdot 2) = f(4) = 3$ | $(2, 3)$ |
| 4 | $(4, 3)$ | 3 | $f(2 \cdot 4) = f(8) =?$ | |
| 5 | $(5, 5)$ | 5 | $f(2 \cdot 5) = f(10) =?$ | |

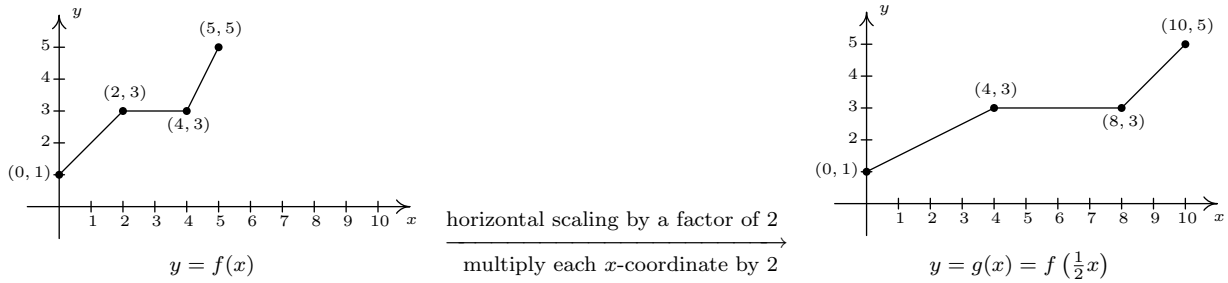
| $x$ | $2x$ | $g(x) = f(2x)$ | $(x, g(x))$ |
|---|---|---|---|
| 0 | 0 | $g(0) = f(0) = 1$ | $(0, 0)$ |
| 1 | 2 | $g(1) = f(2) = 3$ | $(1, 3)$ |
| 2 | 4 | $g(2) = f(4) = 3$ | $(2, 3)$ |
| $\frac{5}{2}$ | 5 | $g\left(\frac{5}{2}\right) = f(5) = 5$ | $\left(\frac{5}{2}, 5\right)$ |

In general, if $(a, b)$ is on the graph of $f$, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of $g$. In other words, to graph $g$ we divide the $x$-coordinates of the

points on the graph of $f$ by 2. This results in a horizontal scaling[9] by a factor of $\frac{1}{2}$.




If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the $x$-coordinates of the points on the graph of $f$ by 2 which results in a horizontal scaling[10] by a factor of 2, as demonstrated below.




We have the following theorem.

THEOREM 2.6. **Horizontal Scalings.** Suppose $f$ is a function and $b > 0$. To graph $y = f(bx)$, **divide** all of the ***x*-coordinates** of the points on the graph of $f$ by $b$. We say the graph of $f$ has been **horizontally scaled** by a factor of $\frac{1}{b}$.

- If $0 < b < 1$, we say the graph of $f$ has undergone a horizontal stretch (expansion, dilation) by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of $f$ has undergone a horizontal shrink (compression, contraction) by a factor of $b$.

Theorem 2.6 tells us that if we multiply the **input** to a function by $b$, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$ since the $x$-values are divided by $b$ to produce corresponding

[9] Also called 'horizontal shrink,''horizontal compression' or 'horizontal contraction' by a factor of 2.
[10] Also called 'horizontal stretch,''horizontal expansion' or 'horizontal dilation' by a factor of 2.

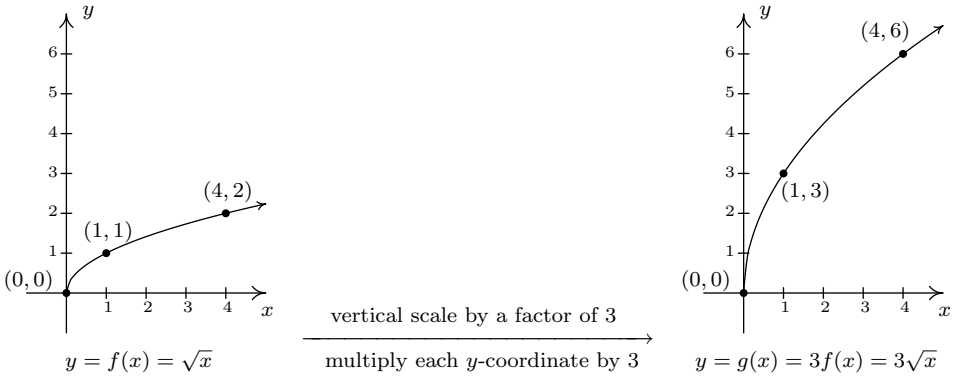
points on the graph of $f(bx)$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

EXAMPLE 2.5.3. Let $f(x) = \sqrt{x}$. Use the graph of $f$ from Example 2.5.1 to graph the following functions below. Also, state their domains and ranges.

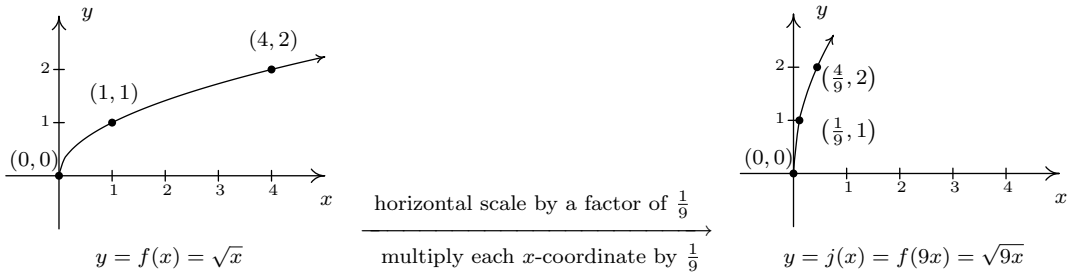
1. $g(x) = 3\sqrt{x}$
2. $j(x) = \sqrt{9x}$
3. $m(x) = 1 - \sqrt{\frac{x+3}{2}}$

SOLUTION.

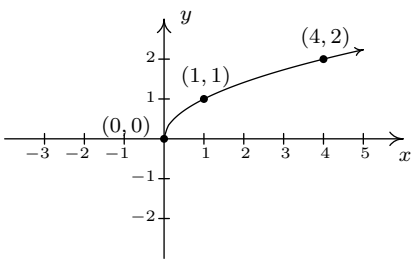
1. First we note that the domain of $g$ is $[0, \infty)$ for the usual reason. Next, we have $g(x) = 3f(x)$ so by Theorem 2.5, we obtain the graph of $g$ by multiplying all of the $y$-coordinates of the points on the graph of $f$ by 3. The result is a vertical scaling of the graph of $f$ by a factor of 3. We find the range of $g$ is also $[0, \infty)$.




2. To determine the domain of $j$, we solve $9x \geq 0$ to find $x \geq 0$. Our domain is once again $[0, \infty)$. We recognize $j(x) = f(9x)$ and by Theorem 2.6, we obtain the graph of $j$ by dividing the $x$-coordinates of the points on the graph of $f$ by 9. From the graph, we see the range of $j$ is also $[0, \infty)$.



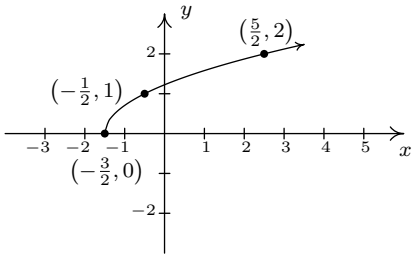
3. Solving $\frac{x+3}{2} \geq 0$ gives $x \geq -3$, so the domain of $m$ is $[-3, \infty)$. To take advantage of what we know of transformations, we rewrite $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$, or $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$. Focusing on the inputs first, we note that the input to $f$ in the formula for $m(x)$ is $\frac{1}{2}x + \frac{3}{2}$. Multiplying the $x$ by $\frac{1}{2}$ corresponds to a horizontal stretch by a factor of 2, and adding the $\frac{3}{2}$ corresponds to a shift to the left by $\frac{3}{2}$. As before, we resolve which to perform first by thinking about how we would find the point on $m$ corresponding to a point on $f$, in this case, $(4, 2)$. To use $f(4) = 2$, we solve $\frac{1}{2}x + \frac{3}{2} = 4$. Our first step is to subtract the $\frac{3}{2}$ (the horizontal shift) to obtain $\frac{1}{2}x = \frac{5}{2}$. Next, we multiply by 2 (the horizontal stretch) and obtain $x = 5$. We define two intermediate functions to handle first the shift, then the stretch. In accordance with Theorem 2.3, $m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$ will shift the graph of $f$ to the left $\frac{3}{2}$ units.




$y = f(x) = \sqrt{x}$

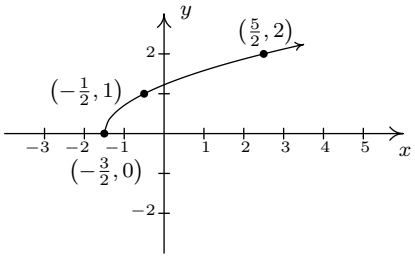
shift left $\frac{3}{2}$ units

subtract $\frac{3}{2}$ from each $x$-coordinate




$y = m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$

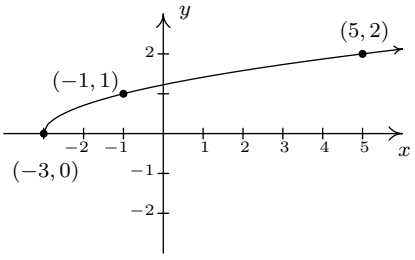
Next, $m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ will, according to Theorem 2.6, horizontally stretch the graph of $m_1$ by a factor of 2.




$y = m_1(x) = \sqrt{x + \frac{3}{2}}$

horizontal scale by a factor of 2

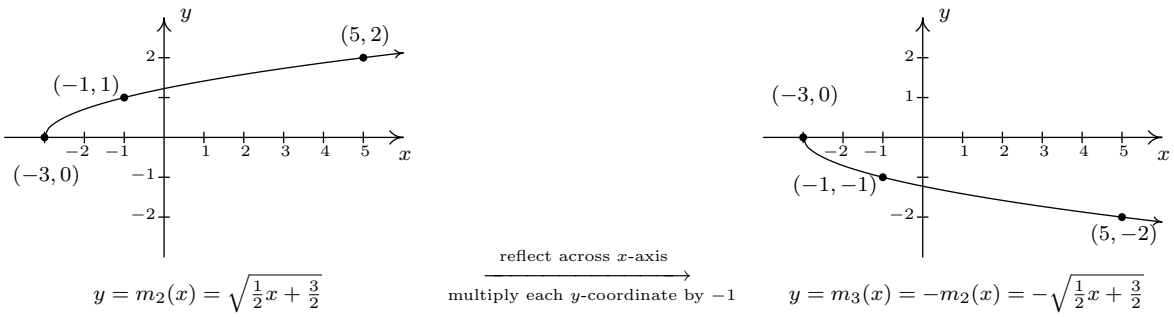
multiply each $x$-coordinate by 2



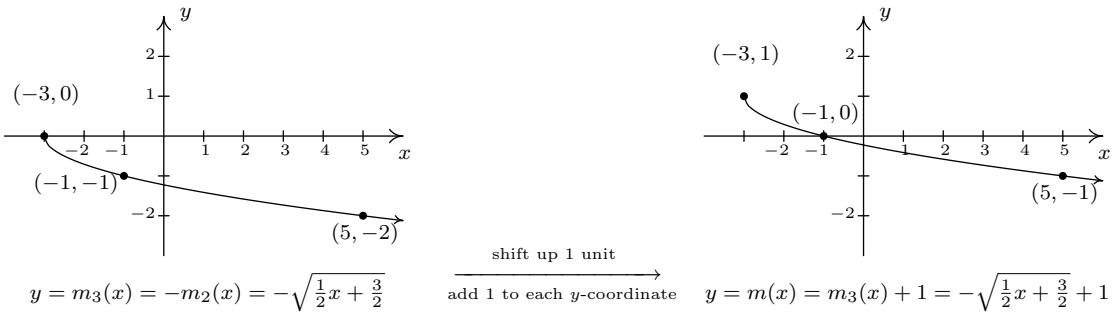

$y = m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

We now examine what's happening to the outputs. From $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$, we see the output from $f$ is being multiplied by $-1$ (a reflection about the $x$-axis) and then a 1 is added (a vertical shift up 1). As before, we can determine the correct order by looking at how the point $(4, 2)$ is moved. We have already determined that to make use of the equation $f(4) = 2$, we need to substitute $x = 5$. We get $m(5) = -f\left(\frac{1}{2}(5) + \frac{3}{2}\right) + 1 = -f(4) + 1 = -2 + 1 = -1$. We see that $f(4)$ (the output from $f$) is first multiplied by $-1$ then the 1 is added meaning we

first reflect the graph about the $x$-axis then shift up 1. Theorem 2.4 tells us $m_3(x) = -m_2(x)$ will handle the reflection.




Finally, to handle the vertical shift, Theorem 2.2 gives $m(x) = m_3(x) + 1$, and we see that the range of $m$ is $(-\infty, 1]$.




□

Some comments about Example 2.5.3 are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions $g$ and $j$ are the same, since $j$ and $g$ have the same domains and $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$. (We invite the reader to verify that the all of the points we plotted on the graph of $g$ lie on the graph of $j$ and vice-versa.) Hence, for $f(x) = \sqrt{x}$, a vertical stretch by a factor of 3 and a horizontal shrink by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function $m$, we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

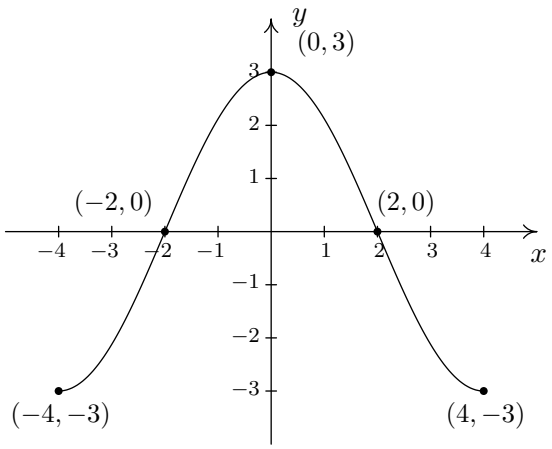
THEOREM 2.7. **Transformations.** Suppose $f$ is a function. To graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract $H$ from each of the $x$-coordinates of the points on the graph of $f$. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Divide the $x$-coordinates of the points on the graph obtained in Step 1 by $B$. This results in a horizontal scaling, but may also include a reflection about the $y$-axis if $B < 0$.
3. Multiply the $y$-coordinates of the points on the graph obtained in Step 2 by $A$. This results in a vertical scaling, but may also include a reflection about the $x$-axis if $A < 0$.
4. Add $K$ to each of the $y$-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Theorem 2.7 can be established by generalizing the techniques developed in this section. Suppose $(a, b)$ is on the graph of $f$. Then $f(a) = b$, and to make good use of this fact, we set $Bx + H = a$ and solve. We first subtract the $H$ (causing the horizontal shift) and then divide by $B$. If $B$ is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{B}$. If $B < 0$, then we have a factor of $-1$ in play, and dividing by it induces a reflection about the $y$-axis. So we have $x = \frac{a-H}{B}$ as the input to $g$ which corresponds to the input $x = a$ to $f$. We now evaluate $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$. We notice that the output from $f$ is first multiplied by $A$. As with the constant $B$, if $A > 0$, this induces only a vertical scaling. If $A < 0$, then the $-1$ induces a reflection across the $x$-axis. Finally, we add $K$ to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem 2.7 is to think of the quantity $Bx + H$ is the 'inside' of the function $f$. What's happening inside $f$ affects the inputs or $x$-coordinates of the points on the graph of $f$. To find the $x$-coordinates of the corresponding points on $g$, we undo what has been done to $x$ in the same way we would solve an equation. What's happening to the output can be thought of as things happening 'outside' the function, $f$. Things happening outside affect the outputs or $y$-coordinates of the points on the graph of $f$. Here, we follow the usual order of operations agreement: we first multiply by $A$ then add $K$ to find the corresponding $y$-coordinates on the graph of $g$.

EXAMPLE 2.5.4. Below is the complete graph of $y = f(x)$. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.




SOLUTION. We use Theorem 2.7 to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of $f$ to their new locations. We first rewrite $g(x)$ in the form presented in Theorem 2.7, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the $x$-coordinates of the key points and solve. For example, solving $-2x+1=-4$, we first subtract 1 to get $-2x=-5$ then divide by $-2$ to get $x=\frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by $-2$ can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by $-1$ which causes a reflection across the $y$-axis. We summarize the results in the table below.

| $(a, f(a))$ | $a$ | $-2x+1=a$ | $x$ |
|---|---|---|---|
| $(-4,-3)$ | $-4$ | $-2x+1=-4$ | $x=\frac{5}{2}$ |
| $(-2,0)$ | $-2$ | $-2x+1=-2$ | $x=\frac{3}{2}$ |
| $(0,3)$ | $0$ | $-2x+1=0$ | $x=\frac{1}{2}$ |
| $(2,0)$ | $2$ | $-2x+1=2$ | $x=-\frac{1}{2}$ |
| $(4,-3)$ | $4$ | $-2x+1=4$ | $x=-\frac{3}{2}$ |

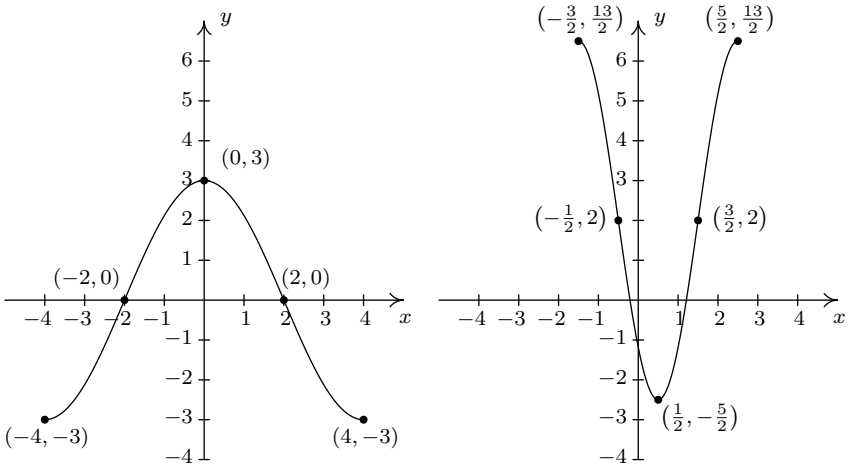
Next, we take each of the $x$ values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding $y$-values. Substituting $x=\frac{5}{2}$, and using the fact that $f(-4)=-3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see the output from $f$ is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by $-1$, we see we have a vertical stretch by a factor of $\frac{3}{2}$ followed by a reflection across the $x$-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below.

| $x$ | $g(x)$ | $(x, g(x))$ |
|---|---|---|
| $\frac{5}{2}$ | $\frac{13}{2}$ | $\left(\frac{5}{2}, \frac{13}{2}\right)$ |
| $\frac{3}{2}$ | $2$ | $\left(\frac{3}{2}, 2\right)$ |
| $\frac{1}{2}$ | $-\frac{5}{2}$ | $\left(\frac{1}{2}, -\frac{5}{2}\right)$ |
| $-\frac{1}{2}$ | $2$ | $\left(-\frac{1}{2}, 2\right)$ |
| $-\frac{3}{2}$ | $\frac{13}{2}$ | $\left(-\frac{3}{2}, \frac{13}{2}\right)$ |

To graph $g$, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting $f$ and $g$ side-by-side gives




The reader is strongly encouraged[11] to graph the series of functions which shows the gradual transformation of the graph of $f$ into the graph of $g$. We have outlined the sequence of transformations in the above exposition; all that remains is to plot all five intermediate stages. □

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

EXAMPLE 2.5.5. Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of $f$ undergoing the following sequence of transformations. Check your answer using a graphing calculator.

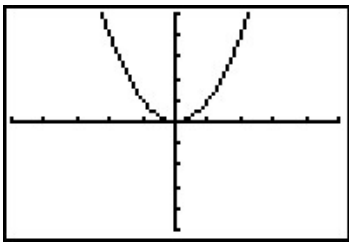
1. Vertical shift up 2 units
2. Reflection across the $x$-axis

[11] You really should do this once in your life.

3. Horizontal shift right 1 unit

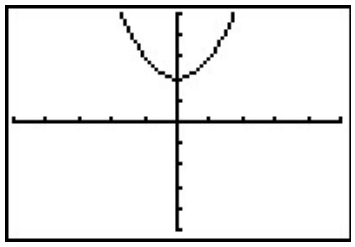
4. Horizontal stretch by a factor of 2

SOLUTION. We build up to a formula for $g(x)$ using intermediate functions as we've seen in previous examples. We let $g_1$ take care of our first step. Theorem 2.2 tells us $g_1(x) = f(x)+2 = x^2+2$. Next, we reflect the graph of $g_1$ about the $x$-axis using Theorem 2.4: $g_2(x) = -g_1(x) = -\left(x^2+2\right) = -x^2-2$. We shift the graph to the right 1 unit, according to Theorem 2.3, by setting $g_3(x) = g_2(x-1) = -(x-1)^2-2 = -x^2+2x-3$. Finally, we induce a horizontal stretch by a factor of 2 using Theorem 2.6 to get $g(x) = g_3\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2+2\left(\frac{1}{2}x\right)-3$ which yields $g(x) = -\frac{1}{4}x^2+x-3$. We use the calculator to graph the stages below to confirm our result.

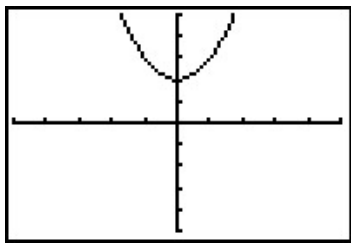


$y = f(x) = x^2$

shift up 2 units
add 2 to each $y$-coordinate

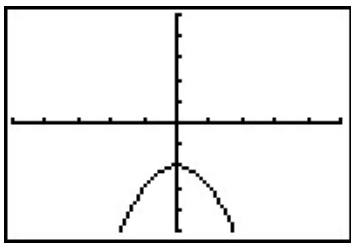


$y = g_1(x) = f(x)+2 = x^2+2$

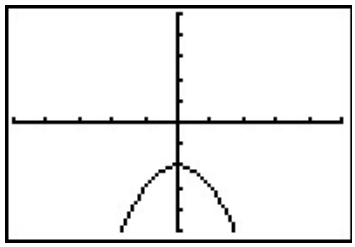


$y = g_1(x) = x^2+2$

reflect across $x$-axis
multiply each $y$-coordinate by $-1$

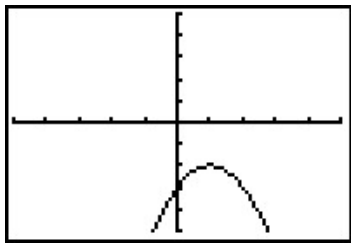


$y = g_2(x) = -g_1(x) = -x^2-2$

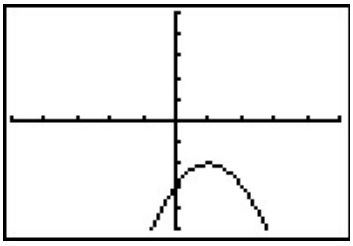


$y = g_2(x) = -x^2-2$

shift right 1 unit
add 1 to each $x$-coordinate

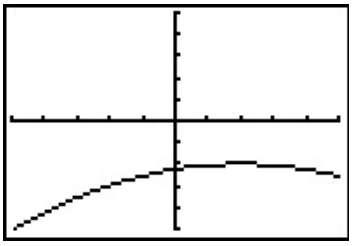


$y = g_3(x) = g_2(x-1) = -x^2+2x-3$



$y = g_3(x) = -x^2 + 2x - 3$

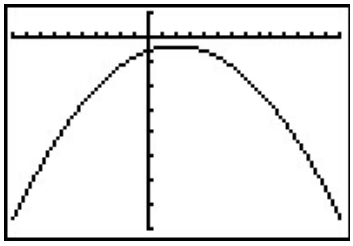
horizontal stretch by a factor of 2
multiply each $x$-coordinate by 2



$y = g(x) = g_3\left(\frac{1}{2}x\right) = -\frac{1}{4}x^2 + x - 3$

□

We have kept the viewing window the same in all of the graphs above. This had the undesirable consequence of making the last graph look 'incomplete' in that we cannot see the original shape of $f(x) = x^2$. Altering the viewing window results in a more complete graph of the transformed function as seen below.

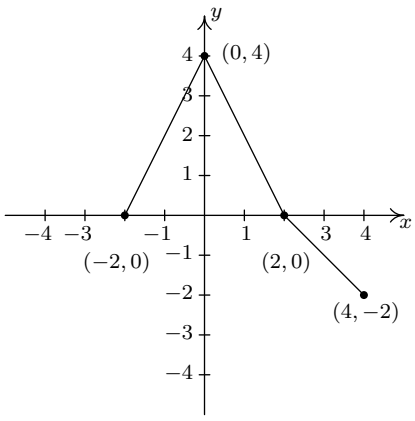


$y = g(x)$

This example brings our first chapter to a close. In the chapters which lie ahead, be on the lookout for the concepts developed here to resurface as we study different families of functions.

### 2.5.1 Exercises

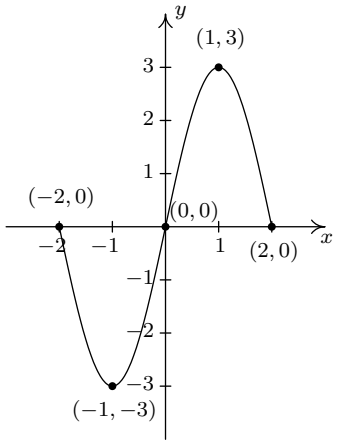
1. The complete graph of $y = f(x)$ is given below. Use it to graph the following functions.




The graph of $y = f(x)$

(a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = \frac{1}{2}f(x)$
(d) $y = f(2x)$
(e) $y = -f(x)$
(f) $y = f(-x)$
(g) $y = f(x + 1) - 1$
(h) $y = 1 - f(x)$
(i) $y = \frac{1}{2}f(x + 1) - 1$

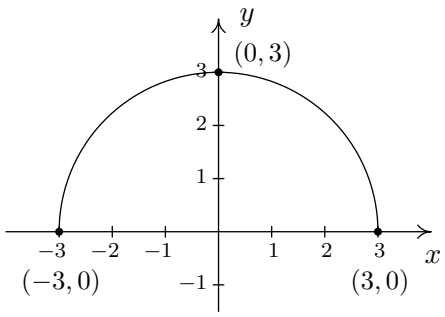
2. The complete graph of $y = S(x)$ is given below. Use it to graph the following functions.




The graph of $y = S(x)$

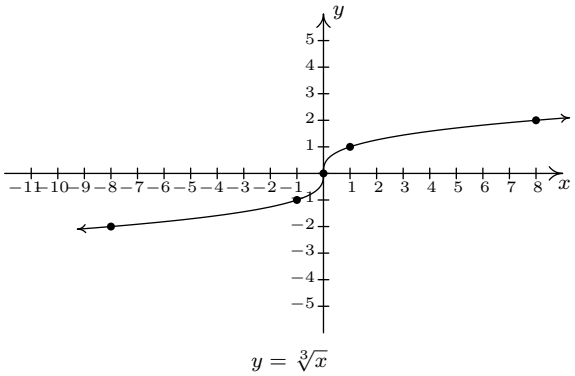
(a) $y = S(x + 1)$
(b) $y = S(-x + 1)$
(c) $y = \frac{1}{2}S(-x + 1)$
(d) $y = \frac{1}{2}S(-x + 1) + 1$

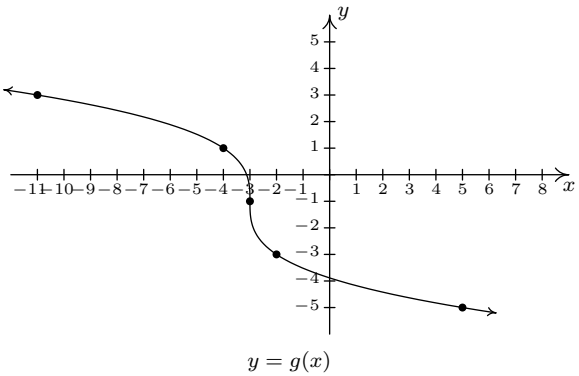
3. The complete graph of $y = f(x)$ is given below. Use it to graph the following functions.




(a) $g(x) = f(x) + 3$
(b) $h(x) = f(x) - \frac{1}{2}$
(c) $j(x) = f\left(x - \frac{2}{3}\right)$
(d) $a(x) = f(x + 4)$
(e) $b(x) = f(x + 1) - 1$
(f) $c(x) = \frac{3}{5}f(x)$
(g) $d(x) = -2f(x)$
(h) $k(x) = f\left(\frac{2}{3}x\right)$
(i) $m(x) = -\frac{1}{4}f(3x)$
(j) $n(x) = 4f(x - 3) - 6$
(k) $p(x) = 4 + f(1 - 2x)$
(l) $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

4. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for $g$ based on transformations of the graph of $f$. Check your answer by confirming that the points shown on the graph of $g$ satisfy the equation $y = g(x)$.



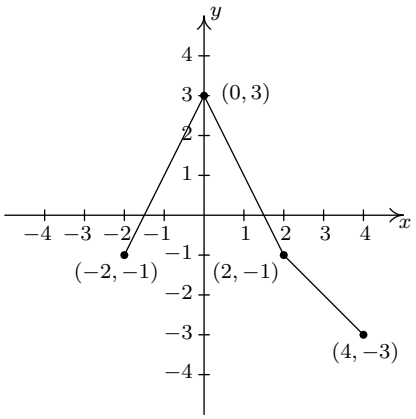




5. For many common functions, the properties of algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x} = 3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$. What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

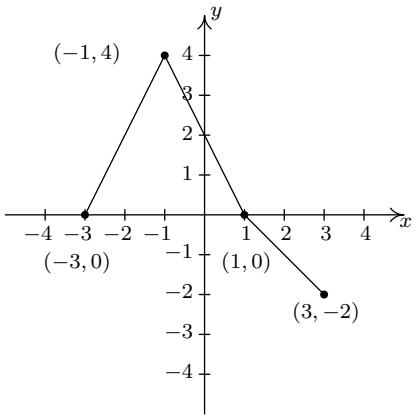
6. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.
7. What happens if you reflect an even function across the $y$-axis?
8. What happens if you reflect an odd function across the $y$-axis?
9. What happens if you reflect an even function across the $x$-axis?
10. What happens if you reflect an odd function across the $x$-axis?
11. How would you describe symmetry about the origin in terms of reflections?
12. As we saw in Example 2.5.5, the viewing window on the graphing calculator affects how we see the transformations done to a graph. Using two different calculators, find viewing windows so that $f(x) = x^2$ on the one calculator looks like $g(x) = 3x^2$ on the other.

### 2.5.2 Answers

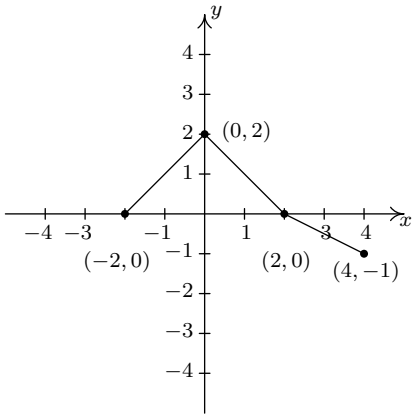
1. (a) $y = f(x) - 1$



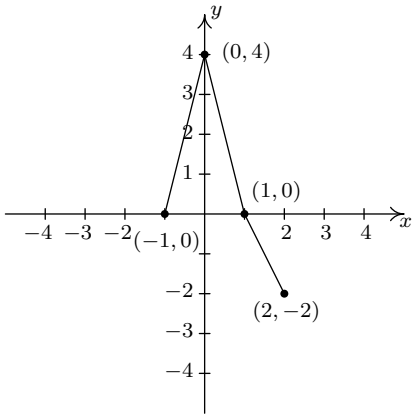

(b) $y = f(x + 1)$



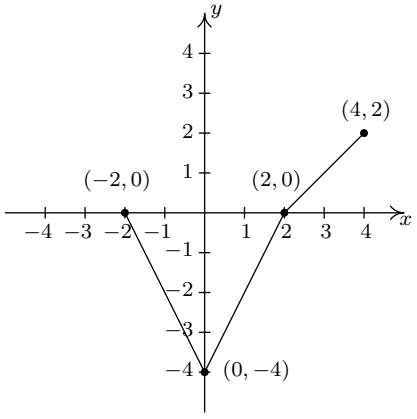

(c) $y = \frac{1}{2} f(x)$



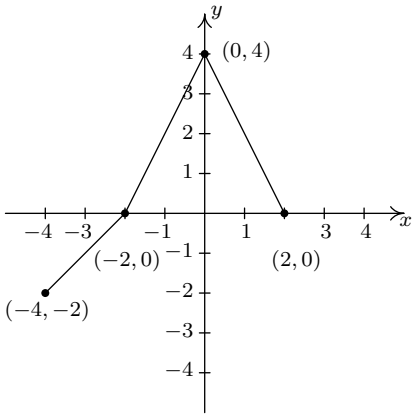

(d) $y = f(2x)$



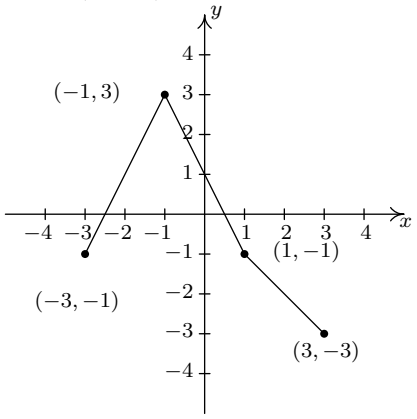

(e) $y = -f(x)$



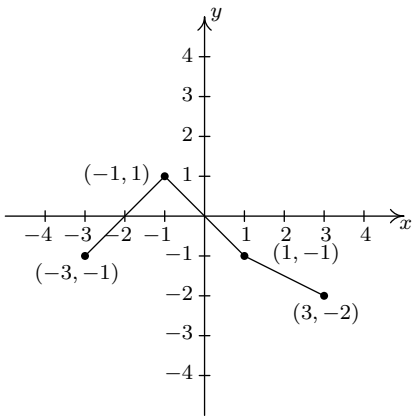

(f) $y = f(-x)$



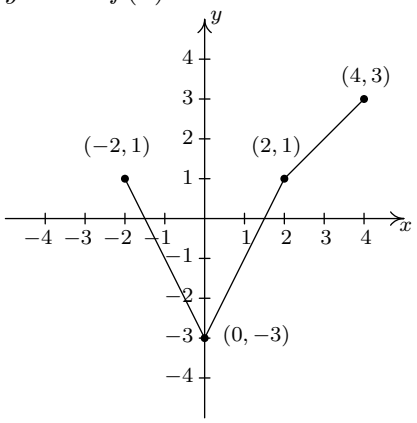
(g) $y = f(x+1) - 1$



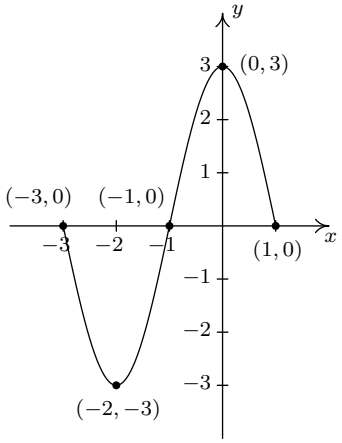

(i) $y = \frac{1}{2}f(x+1) - 1$



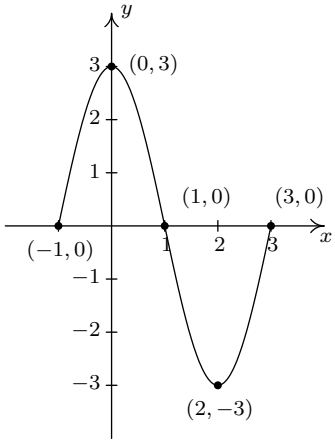

(h) $y = 1 - f(x)$



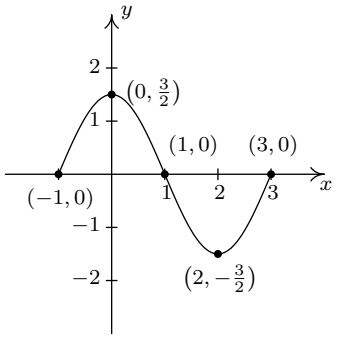

2. (a) $y = S(x+1)$



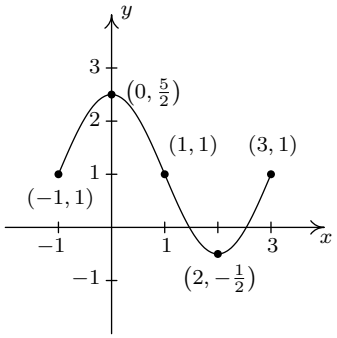

(b) $y = S(-x+1)$



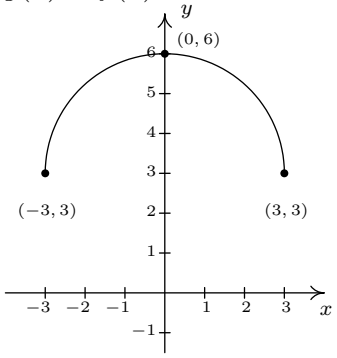
(c) $y = \frac{1}{2}S(-x+1)$



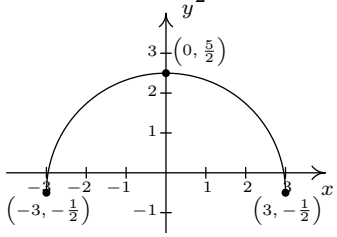

(d) $y = \frac{1}{2}S(-x+1)+1$



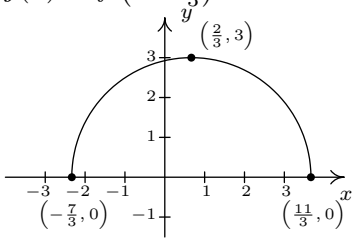

3. (a) $g(x) = f(x) + 3$



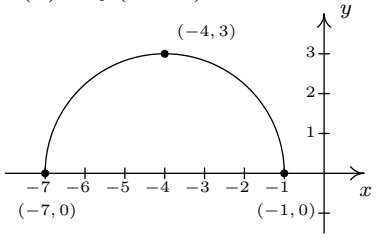

(b) $h(x) = f(x) - \frac{1}{2}$



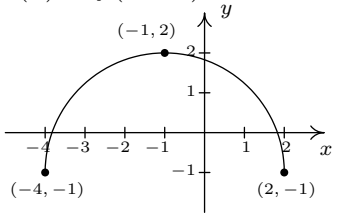

(c) $j(x) = f\left(x - \frac{2}{3}\right)$



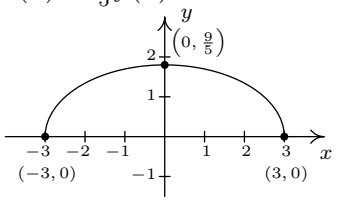

(d) $a(x) = f(x+4)$



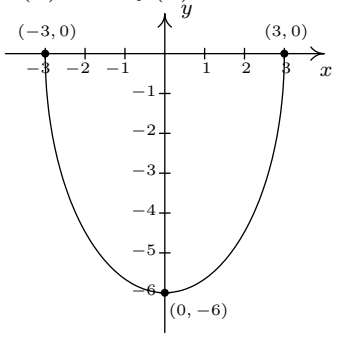

(e) $b(x) = f(x+1) - 1$



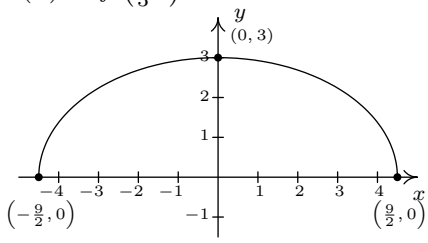

(f) $c(x) = \frac{3}{5}f(x)$



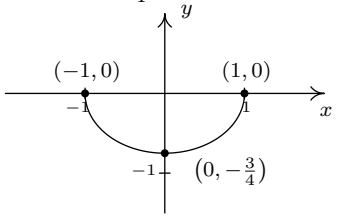
(g) $d(x) = -2f(x)$



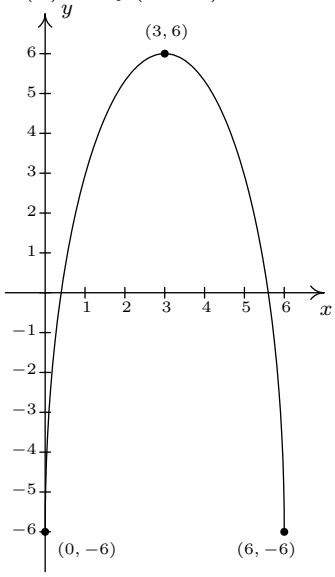

(h) $k(x) = f\left(\frac{2}{3}x\right)$



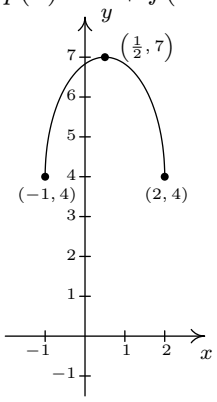

(i) $m(x) = -\frac{1}{4}f(3x)$



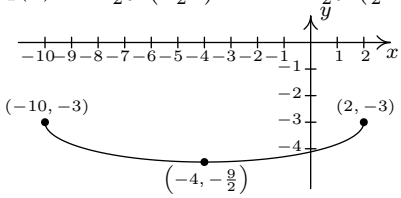

(j) $n(x) = 4f(x-3) - 6$




(k) $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$




(l) $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x + 2\right) - 3$



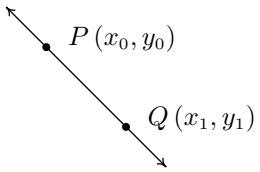

4. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

# Chapter 3

# Linear and Quadratic Functions

## 3.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated below.




To give a sense of the 'steepness' of the line, we recall we can compute the **slope** of the line using the formula below.

Equation 3.1. The **slope** $m$ of the line containing the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Equation 3.1 are in order. First, don't ask why we use the letter '$m$' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

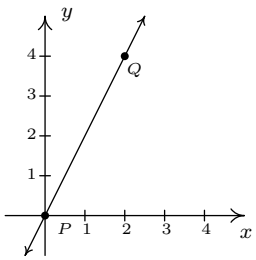
[1] See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

EXAMPLE 3.1.1. Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

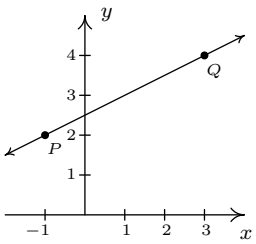
1. $P(0,0)$, $Q(2,4)$
2. $P(-1,2)$, $Q(3,4)$
3. $P(-2,3)$, $Q(2,-3)$
4. $P(-3,2)$, $Q(4,2)$
5. $P(2,3)$, $Q(2,-1)$
6. $P(2,3)$, $Q(2.1,-1)$

SOLUTION. In each of these examples, we apply the slope formula, Equation 3.1.

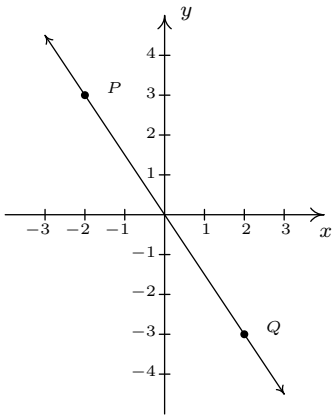
1. $m = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$



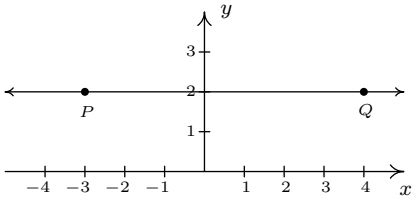

2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



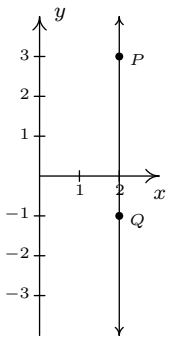

3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



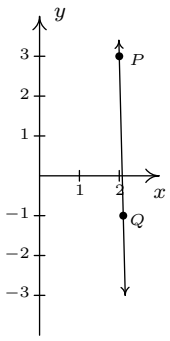

4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined

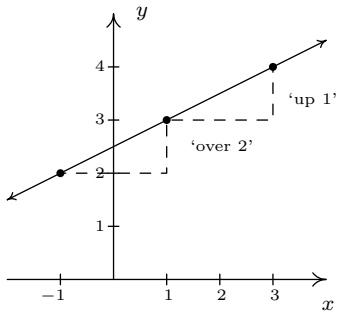



6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$




□

A few comments about Example 3.1.1 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a vertical line.[2] Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 3.1.1, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown below.




[2]Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

Using more formal notation, given points $(x_0, y_0)$ and $(x_1, y_1)$, we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'. Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of $y$ with respect to $x$. Rates of change abound in the 'real world,' as the next example illustrates.

EXAMPLE 3.1.2. At 6 AM, it is 24°F; at 10 AM, it is 32°F.

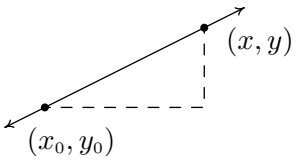
1. Find the slope of the line containing the points $(6, 24)$ and $(10, 32)$.
2. Interpret your answer to the first part in terms of temperature and time.
3. Predict the temperature at noon.

SOLUTION.

1. For the slope, we have $m = \frac{32-24}{10-6} = \frac{8}{4} = 2$.
2. Since the values in the numerator correspond to the temperatures in °F, and the values in the denominator correspond to time in hours, we can interpret the slope as $2 = \frac{2}{1} = \frac{2^\circ\,\text{F}}{1\,\text{hour}}$, or 2°F per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of 2°F per hour.
3. Noon is two hours after 10 AM. Assuming a temperature increase of 2°F per hour, in two hours the temperature should rise 4°F. Since the temperature at 10 AM is 32°F, we would expect the temperature at noon to be $32 + 4 = 36$°F.

Now it may well happen that in the previous scenario, at noon the temperature is only 33°F. This doesn't mean our calculations are incorrect. Rather, it means that the temperature change throughout the day isn't a constant 2°F per hour. Mathematics is often used to describe, or **model**, real world phenomena. Mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world. □

In Section 1.2, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of $m$ and contains the point $(x_0, y_0)$. Suppose $(x, y)$ is another point on the line, as indicated below.



We have

$$\begin{aligned} m &= \frac{y-y_0}{x-x_0} \\ m(x-x_0) &= y-y_0 \\ y-y_0 &= m(x-x_0) \\ y &= m(x-x_0)+y_0. \end{aligned}$$

We have just derived the **point-slope form** of a line.[3]

Equation 3.2. The **point-slope form** of the line with slope $m$ containing the point $(x_0, y_0)$ is the equation $y = m(x-x_0)+y_0$.

Example 3.1.3. Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution. In order to use Equation 3.2 we need to find the slope of the line in question. So we use Equation 3.1 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point $(x_0, y_0)$. We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y &= m(x-x_0)+y_0 \\ y &= -\frac{2}{3}(x-(-1))+3 \\ y &= -\frac{2}{3}x-\frac{2}{3}+3 \\ y &= -\frac{2}{3}x+\frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x+\frac{7}{3}$ algebraically, as we did in Section 1.3.

□

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in

[3] We can also understand this equation in terms of applying transformations to the function $I(x) = x$. See the exercises.

Intermediate Algebra. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the $y$-intercept of the line $y = mx + b$ is $(0, b)$.

EQUATION 3.3. The **slope-intercept form** of the line with slope $m$ and $y$-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$ which matches our formula for a horizontal line given in Section 1.2. The formula given in Equation 3.3 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (why?) and so we have finally reached a good point to introduce **linear functions**.

DEFINITION 3.1. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where $m$ and $b$ are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

DEFINITION 3.2. A **constant function** is a function of the form

$$f(x) = b,$$

where $b$ is real number. The domain of a constant function is $(-\infty, \infty)$.

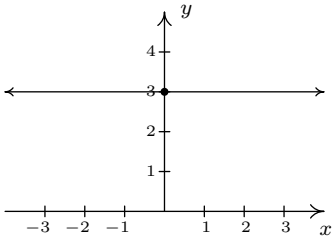
Recall that to graph a function, $f$, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope $m$ and $y$-intercept $(0, b)$; the graph of a constant function is a horizontal line (with slope $m = 0$) and a $y$-intercept of $(0, b)$. Now think back to Section 2.4.1, specifically Definition 2.5 concerning increasing, decreasing and constant functions. A line with positive slope was called an increasing line because a linear function with $m > 0$ is an increasing function. Similarly, a line with a negative slope was called a decreasing line because a linear function with $m < 0$ is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

EXAMPLE 3.1.4. Graph the following functions. Identify the slope and $y$-intercept.

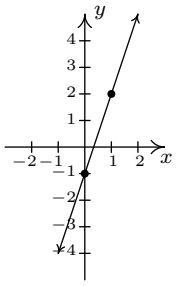
1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3 - 2x}{4}$
4. $f(x) = \dfrac{x^2 - 4}{x - 2}$

Solution.

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$.
2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 3.3 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our $y$-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$.



$f(x) = 3$

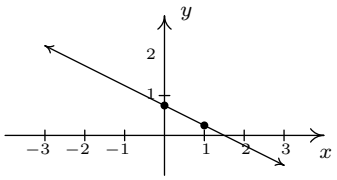


$f(x) = 3x - 1$

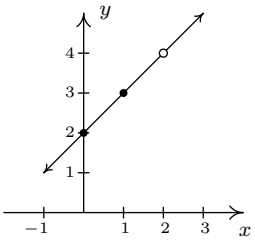
3. At first glance, the function $f(x) = \frac{3-2x}{4}$ does not fit the form in Definition 3.1 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} = -\frac{1}{2}x + \frac{3}{4}$. We identify $m = -\frac{1}{2}$ and $b = \frac{3}{4}$. Hence, our graph is a line with a slope of $-\frac{1}{2}$ and a $y$-intercept of $\left(0, \frac{3}{4}\right)$. Plotting an additional point, we can choose $(1, f(1))$ to get $\left(1, \frac{1}{4}\right)$.
4. If we simplify the expression for $f$, we get

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{\cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}} = x + 2.$$

If we were to state $f(x) = x + 2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so before we simplify! In this case, $f$ has big problems when $x = 2$, and as such, the domain of $f$ is $(-\infty, 2) \cup (2, \infty)$. To indicate this, we write $f(x) = x + 2$, $x \neq 2$. So, except at $x = 2$, we graph the line $y = x + 2$. The slope $m = 1$ and the $y$-intercept is $(0, 2)$. A second point on the graph is $(1, f(1)) = (1, 3)$. Since our function $f$ is not defined at $x = 2$, we put an open circle at the point that would be on the line $y = x + 2$ when $x = 2$, namely $(2, 4)$.




$$f(x) = \frac{3 - 2x}{4}$$




$$f(x) = \frac{x^2 - 4}{x - 2}$$

□

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 3.1, since some algebraic manipulations may be needed to rewrite a given function to match 'the form.' Keep in mind that the domains of linear and constant functions are all real numbers, $(-\infty, \infty)$, and so while $f(x) = \frac{x^2-4}{x-2}$ simplified to a formula $f(x) = x + 2$, $f$ is not considered a linear function since its domain excludes $x = 2$. However, we would consider

$$f(x) = \frac{2x^2 + 2}{x^2 + 1}$$

to be a constant function since its domain is all real numbers (why?) and

$$f(x) = \frac{2x^2 + 2}{x^2 + 1} = \frac{2\cancel{(x^2 + 1)}}{\cancel{(x^2 + 1)}} = 2$$

The following example uses linear functions to model some basic economic relationships.

EXAMPLE 3.1.5. The cost, $C$, in dollars, to produce $x$ PortaBoy game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(10)$.
2. How many PortaBoys can be produced for \$15,000?
3. Explain the significance of the restriction on the domain, $x \geq 0$.
4. Find and interpret $C(0)$.
5. Find and interpret the slope of the graph of $y = C(x)$.

SOLUTION.

1. To find $C(10)$, we replace every occurrence of $x$ with 10 in the formula for $C(x)$ to get $C(10) = 80(10) + 150 = 950$. Since $x$ represents the number of PortaBoys produced, and $C(x)$ represents the cost, in dollars, $C(10) = 950$ means it costs \$950 to produce 10 PortaBoys for the local retailer.

2. To find how many PortaBoys can be produced for \$15,000, we set the cost, $C(x)$, equal to 15000, and solve for $x$

$$\begin{aligned} C(x) &= 15000 \\ 80x + 150 &= 15000 \\ 80x &= 14850 \\ x &= \frac{14850}{80} = 185.625 \end{aligned}$$

   Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for \$15,000.

3. The restriction $x \geq 0$ is the applied domain, as discussed in Section 2.2. In this context, $x$ represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.[4]

4. To find $C(0)$, we replace every occurrence of $x$ with 0 in the formula for $C(x)$ to get $C(0) = 80(0) + 150 = 150$. This means it costs \$150 to produce 0 PortaBoys. The \$150 is often called the **fixed** or **start-up** cost of this venture. (What might contribute to this cost?)

5. If we were to graph $y = C(x)$, we would be graphing the portion of the line $y = 80x + 150$ for $x \geq 0$. We recognize the slope, $m = 80$. Like any slope, we can interpret this as a rate of change. In this case, $C(x)$ is the cost in dollars, while $x$ measures the number of PortaBoys so

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1\,\text{PortaBoy}}.$$

   In other words, the cost is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for this venture. □

The next example asks us to find a linear function to model a related economic problem.

Example 3.1.6. The local retailer in Example 3.1.5 has determined that the number of PortaBoy game systems sold in a week, $x$, is related to the price of each system, $p$, in dollars. When the price was \$220, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at \$190 a piece.

1. Find a linear function which fits this data. Use the weekly sales, $x$, as the independent variable and the price $p$, as the dependent variable.

2. Find a suitable applied domain.

[4] Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgivable in some textbooks but not to us.

3. Interpret the slope.
4. If the retailer wants to sell 150 PortaBoys next week, what should the price be?
5. What would the weekly sales be if the price were set at \$150 per system?

SOLUTION.

1. We recall from Section 2.2 the meaning of 'independent' and 'dependent' variable. Since $x$ is to be the independent variable, and $p$ the dependent variable, we treat $x$ as the input variable and $p$ as the output variable. Hence, we are looking for a function of the form $p(x) = mx + b$. To determine $m$ and $b$, we use the fact that 20 PortaBoys were sold during the week the price was 220 dollars and 40 units were sold when the price was 190 dollars. Using function notation, these two facts can be translated as $p(20) = 220$ and $p(40) = 190$. Since $m$ represents the rate of change of $p$ with respect to $x$, we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

   We now have determined $p(x) = -1.5x + b$. To determine $b$, we can use our given data again. Using $p(20) = 220$, we substitute $x = 20$ into $p(x) = 1.5x + b$ and set the result equal to 220: $-1.5(20) + b = 220$. Solving, we get $b = 250$. Hence, we get $p(x) = -1.5x + 250$. We can check our formula by computing $p(20)$ and $p(40)$ to see if we get 220 and 190, respectively. Incidentally, this equation is sometimes called the **price-demand**[5] equation for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so $x \geq 0$. However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Another constraint, then, is that the price, $p(x) \geq 0$. Solving $-1.5x + 250 \geq 0$ results in $-1.5x \geq -250$ or $x \leq \frac{500}{3} = 166.\overline{6}$. Since $x$ represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for $p$ is $[0, 166]$.

3. The slope $m = -1.5$, once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

4. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150)+250 = 25$. That is, the price would have to be \$25.

5. If the price of a PortaBoy were set at \$150, we have $p(x) = 150$, or, $-1.5x+250 = 150$. Solving, we get $-1.5x = -100$ or $x = 66.\overline{6}$. This means you would be able to sell 66 PortaBoys a week if the price were \$150 per system. □

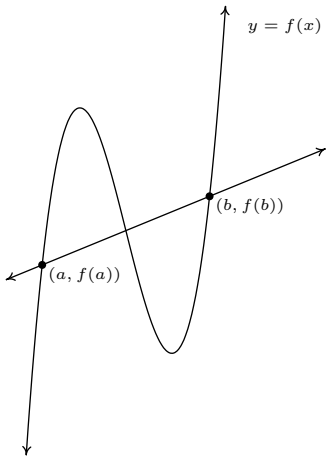
[5] Or simply the **demand** equation

Not all real-world phenomena can be modeled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following:

Definition 3.3. Let $f$ be a function defined on the interval $[a, b]$. The **average rate of change** of $f$ over $[a, b]$ is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of $y = f(x)$, the average rate of change over $[a, b]$ is the slope of the line which connects $(a, f(a))$ and $(b, f(b))$. This is called the **secant line** through these points. For that reason, some textbooks use the notation $m_{\text{sec}}$ for the average rate of change of a function. Note that for a linear function $m = m_{\text{sec}}$, or in other words, its rate of change over an interval is the same as its average rate of change.




The graph of $y = f(x)$ and its secant line through $(a, f(a))$ and $(b, f(b))$

The interested reader may question the adjective 'average' in the phrase 'average rate of change.' In the figure above, we can see that the function changes wildly on $[a, b]$, yet the slope of the secant line only captures a snapshot of the action at $a$ and $b$. This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 miles. Your average speed is $\frac{100\,\text{miles}}{2\,\text{hours}} = 50\,\text{miles per hour}$. However, it is entirely possible that at the start of your journey, you traveled 25 miles per hour, then sped up to 65 miles per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rates of change**, and this is one of the central themes of Calculus.[6]

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

[6] Here we go again...

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

EXAMPLE 3.1.7. The **revenue** of selling $x$ units at a price $p$ per unit is given by the formula $R = xp$. Suppose we are in the scenario of Examples 3.1.5 and 3.1.6.

1. Find and simplify an expression for the weekly revenue $R$ as a function of weekly sales, $x$.
2. Find and interpret the average rate of change of $R$ over the interval $[0, 50]$.
3. Find and interpret the average rate of change of $R$ as $x$ changes from 50 to 100 and compare that to your result in part 2.
4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

SOLUTION.

1. Since $R = xp$, we substitute $p(x) = -1.5x + 250$ from Example 3.1.6 to get
$$R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$$
2. Using Definition 3.3, we get the average rate of change is
$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$
Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases \$175.
3. The wording of this part is slightly different than that in Definition 3.3, but its meaning is to find the average rate of change of $R$ over the interval $[50, 100]$. To find this rate of change, we compute
$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$
In other words, for each additional PortaBoy sold, the revenue increases by \$25. Note while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)
4. Translating the English to the mathematics, we are being asked to find the average rate of change of $R$ over the interval $[100, 150]$. We find
$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$
This means that we are losing \$125 dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?)

□

We close this section with a new look at difference quotients, first introduced in Section 2.2. If we wish to compute the average rate of change of a function $f$ over the interval $[x, x+h]$, then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus.[7]

[7] So, we are not torturing you with these for nothing.

### 3.1.1 EXERCISES

1. Find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

   (a) $m = \frac{1}{7}$, $P(-1, 4)$
   (b) $m = -\sqrt{2}$, $P(0, -3)$
   (c) $m = -5$, $P(\sqrt{3}, 2\sqrt{3})$
   (d) $m = 678$, $P(-1, -12)$

2. Find the slope-intercept form of the line which passes through the given points.

   (a) $P(0, 0)$, $Q(-3, 5)$
   (b) $P(-1, -2)$, $Q(3, -2)$
   (c) $P(5, 0)$, $Q(0, -8)$
   (d) $P(3, -5)$, $Q(7, 4)$

3. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

   (a) Find a linear function $F$ that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert 20°C into Fahrenheit.
   (b) Find a linear function $C$ that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert 110°F into Celsius.
   (c) Is there a temperature $n$ such that $F(n) = C(n)$?

4. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of $x$ dollars. Find a linear function that represents her total weekly pay in terms of $x$. What must her weekly sales be in order for her to earn \$475.00 for the week?

5. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

6. Economic forces beyond anyone's control have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 3.1.5 with this new cost function.

7. In response to the economic forces in the exercise above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Examples 3.1.6 and 3.1.7 with this new data. What difficulties do you encounter?

8. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40°F$ outside and only 5 times per hour if it's $70°F$. Assuming that the number of howls per hour, $N$, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20°F$ outside. What is the applied domain of this function? Why?

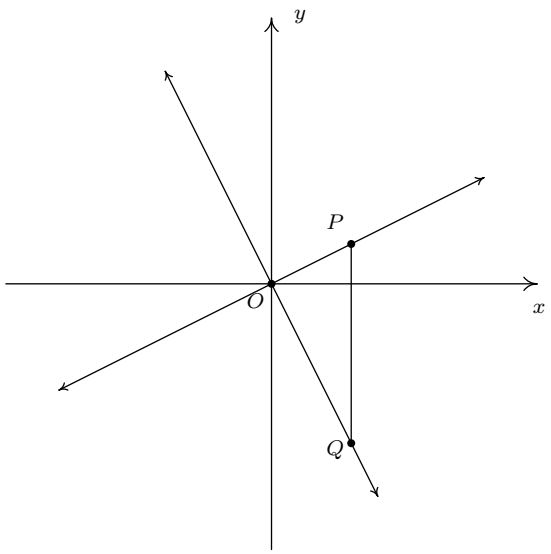
9. (Parallel Lines) Recall from Intermediate Algebra that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In the exercises below, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

(a) $y = 3x + 2,\ P(0, 0)$ (b) $y = -6x + 5,\ P(3, 2)$

10. (Perpendicular Lines) Recall from Intermediate Algebra that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope $m_1$ and the other has slope $m_2$ then $m_1 \cdot m_2 = -1$. (You will be guided through a proof of this result in the next exercise.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume $m_1 \neq 0$ and $m_2 \neq 0$. In the exercises below, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

(a) $y = \frac{1}{3}x + 2,\ P(0, 0)$ (b) $y = -6x + 5,\ P(3, 2)$

11. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0, 0)$ , $P(1, m_1)$ and $Q(1, m_2)$ gives us the following set up.




The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let $d_1$ be the distance from $O$ to $P$, let $d_2$ be the distance from $O$ to $Q$ and let $d_3$ be the distance from $P$ to $Q$. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

12. The function defined by $I(x) = x$ is called the Identity Function.

    (a) Discuss with your classmates why this name makes sense.

    (b) Show that the point-slope form of a line (Equation 3.2) can be obtained from $I$ using a sequence of the transformations defined in Section 2.5.

13. Compute the average rate of change of the given function over the specified interval.

    (a) $f(x) = x^3,\ [-1, 2]$

    (b) $f(x) = \dfrac{1}{x},\ [1, 5]$

    (c) $f(x) = \sqrt{x},\ [0, 16]$

    (d) $f(x) = x^2,\ [-3, 3]$

    (e) $f(x) = \dfrac{x+4}{x-3},\ [5, 7]$

    (f) $f(x) = 3x^2 + 2x - 7,\ [-4, 2]$

14. Compute the average rate of change of the given function over the interval $[x, x + h]$. Here we assume $[x, x + h]$ is in the domain of each function.

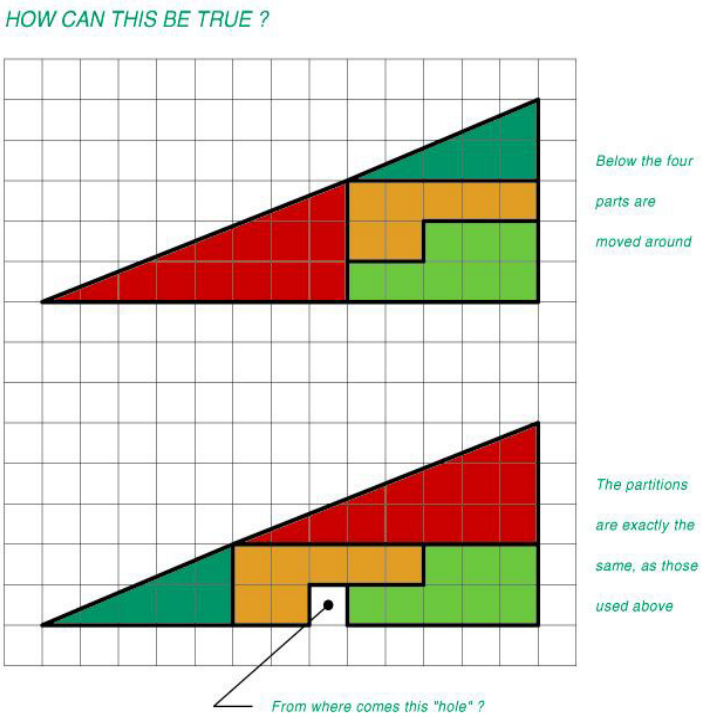
    (a) $f(x) = x^3$

    (b) $f(x) = \dfrac{1}{x}$

    (c) $f(x) = \dfrac{x+4}{x-3}$

    (d) $f(x) = 3x^2 + 2x - 7$

15. Explain:



### 3.1.2 Answers

1.

(a) $y - 4 = \frac{1}{7}(x + 1)$
$y = \frac{1}{7}x + \frac{29}{7}$

(b) $y + 3 = -\sqrt{2}(x - 0)$
$y = -\sqrt{2}x - 3$

(c) $y - 2\sqrt{3} = -5(x - \sqrt{3})$
$y = -5x + 7\sqrt{3}$

(d) $y + 12 = 678(x + 1)$
$y = 678x + 666$

2. (a) $y = -\frac{5}{3}x$
(b) $y = -2$
(c) $y = \frac{8}{5}x - 8$
(d) $y = \frac{9}{4}x - \frac{47}{4}$

3. (a) $F(C) = \frac{9}{5}C + 32$
(b) $C(F) = \frac{5}{9}F - \frac{160}{9}$
(c) $F(-40) = -40 = C(-40)$.

4. $W(x) = 200 + .05x$, She must make \$5500 in weekly sales.

5. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

8. $N(T) = -\frac{2}{15}T + \frac{43}{3}$

Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of $107.5°$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-60°$ so that the applied domain is $[-60, 107.5]$.

9. (a) $y = 3x$
(b) $y = -6x + 20$

10. (a) $y = -3x$
(b) $y = \frac{1}{6}x + \frac{3}{2}$

13. (a) $\dfrac{2^3 - (-1)^3}{2 - (-1)} = 3$
(b) $\dfrac{\frac{1}{5} - \frac{1}{1}}{5 - 1} = -\dfrac{1}{5}$
(c) $\dfrac{\sqrt{16} - \sqrt{0}}{16 - 0} = \dfrac{1}{4}$
(d) $\dfrac{3^2 - (-3)^2}{3 - (-3)} = 0$
(e) $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7 - 5} = -\dfrac{7}{8}$
(f) $\dfrac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2 - (-4)} = -4$

14. (a) $3x^2 + 3xh + h^2$
(b) $\dfrac{-1}{x(x + h)}$
(c) $\dfrac{-7}{(x - 3)(x + h - 3)}$
(d) $6x + 3h + 2$

## 3.2 Defining Functions (Word Problems)

Example 3.2.1. Suppose you have two numbers which add to 12.

**(i)** If one of the numbers is 2, what is the sum of the squares of the two numbers?

Solution: If one of the numbers is 2, then the other number must be $12 - 2 = 10$. Each of their squares would be $2^2 = 4$ and $10^2 = 100$. So the sum of their squares would be $2^2 + 10^2$, which is 104.

**(ii)** If one of the numbers is $x$, then find a function $f(x)$ for the sum of the squares of the two numbers.

Solution: If one of the numbers is $x$, then the other number must be $12-x$. Each of their squares would be $x^2$ and $(12-x)^2$. So the sum of their squares would be $f(x) = x^2 + (12-x)^2$.

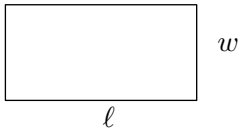
Example 3.2.2. The sum of two numbers is 12. Find a function $f(x)$ which computes the sum of the cubes of the two numbers, where $x$ is one of the two numbers.

Solution: If $x$ is one of two numbers which add to 12, then the other number must be $12-x$. So their cubes would be $x^3$ and $(12-x)^3$, making

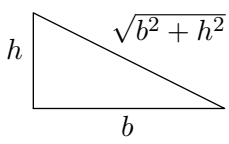
$$f(x) = x^3 + (12-x)^3.$$

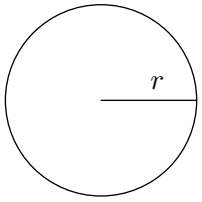
To find a function it helps to remember the following facts:

- The distance between $(x_1, y_1)$ and $(x_2, y_2)$ is $\boxed{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.}$
- The midpoint between $(x_1, y_1)$ and $(x_2, y_2)$ is $\boxed{\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).}$
- A rectangle with dimensions $l$ and $w$ has area $\boxed{l \cdot w}$ and perimeter $\boxed{2l + 2w}$.




- A right triangle with dimensions $b$ and $h$ has area $\boxed{\frac{1}{2}bh}$ and hypothenuse $\boxed{\sqrt{b^2+h^2}}$






▸ A circle with radius $r$ has area $\boxed{\pi r^2}$

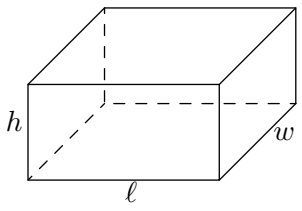
and circumference $\boxed{2\pi r}$

▸ A rectangular box with dimensions $l$, $w$, and $h$

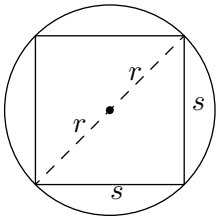
has volume $\boxed{l \cdot w \cdot h}$

and surface area $\boxed{2lw + 2lh + 2hw}$.

The surface area with no top is $\boxed{lw + 2lh + 2wh}$.




EXAMPLE 3.2.3. A square of side $s$ is inscribed inside a circle of radius $r$.




**(i)** Find a formula for the diameter of the circle in terms of $r$.

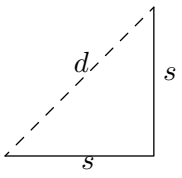
SOLUTION: The diameter of a circle is twice the length of the radius. So $d = 2 \cdot r$.

**(i)** Find a formula for the length of the side of the square in terms of the diameter, then the radius.

SOLUTION: Using the Pythagorean theorem, we find that

$$d^2 = s^2 + s^2 = 2s^2$$

We can solve for the side length to get




$$\begin{aligned} s^2 &= \frac{d^2}{2} \\ \sqrt{s^2} &= \sqrt{\frac{d^2}{2}} = \frac{\sqrt{d^2}}{\sqrt{2}} = \frac{d}{\sqrt{2}} \\ s &= \frac{d}{\sqrt{2}} \end{aligned}$$

(Note that $\sqrt{s^2} = s$ and $\sqrt{d^2} = d$ because $s$ and $d$ represent distances and distances are always positive) Now, we can find the side length in terms of the radius:

$$s = \frac{d}{\sqrt{2}} = \frac{(2r)}{\sqrt{2}} = \frac{(2r)}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2r\sqrt{2}}{2} = r\sqrt{2}.$$

**(iv)** Find a function $P(r)$ which computes the perimeter of the inscribed square as a function of the radius $r$.

Solution: The perimeter of a square is $P = 4s$, and using what we found $s$ to be above, we get

$$P(r) = 4s = 4(r\sqrt{2}) = 4\sqrt{2} \cdot r.$$

**(v)** Find a function $A(r)$ which computes the area of the inscribed square as a function of the radius $r$.

Solution: The area of a square is $A = s^2$, and using what we found $s$ to be above, we get

$$A(s) = s^2 = (r\sqrt{2})^2 = r^2(\sqrt{2})^2 = 2r^2$$

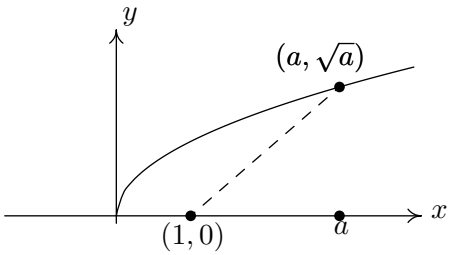
Example 3.2.4. Let $f(x) = \sqrt{x}$.

**(i)** Find the function $d(a)$, which computes the distance from the point $(1, 0)$ to the point on the graph of $y = f(x)$ whose $x$-coordinate is $a$.

Solution: To find this function, we need to convert this statement into a mathematical expression, so

$$d(a) = \begin{cases} \text{the distance from the point } (1,0) \text{ to the point} \\ \text{on the graph of } y = f(x) \text{ whose } x\text{-coordinate is } a. \end{cases}$$

To find the distance between two points, we need to know the coordinates of the two points, then we can use the distance formula. We know one point is $(1, 0)$. The other point is on the graph $y = f(x) = \sqrt{x}$ and has $x$-coordinate equalling $a$. Then the $y$-coordinate for that point would be $y = f(a) = \sqrt{a}$, making the second point be $(a, \sqrt{a})$.



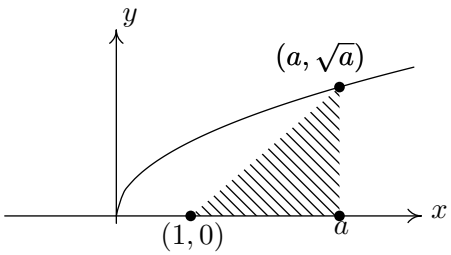

Now, our function is

$$d(a) = \text{the distance from the point } (1,0) \text{ to the point } (a, \sqrt{a}).$$

We can use the distance formula to express this distance and get that

$$d(a) = \sqrt{(1-a)^2 + (0-\sqrt{a})^2} = \sqrt{(1-a)^2 + a} = \sqrt{1 - 2a + a^2 + a} = \sqrt{1 - a + a^2}.$$

**(ii)** Find a function $A(a)$ which computes the area of a triangle whose vertices are $(0, 0)$, $(a, 0)$, and $(a, b)$, where $(a, b)$ is on the graph of $y = f(x)$.




SOLUTION: To compute the area of a triangle, we must use the formula

$$A = \frac{1}{2}(\text{base}) \times (\text{height})$$

Using the picture, we can determine that

$$\text{base } = a, \text{ height } = b$$

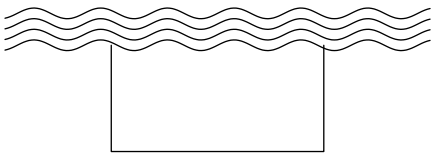
So the formula for the area is

$$A = \frac{1}{2}ab.$$

Now, we need to reduce this formula to a single variable $a$. So we must find a relationship between $a$ and $b$ (an equation) which we can solve for $b$ and substitute. As in part (i) the point $(a, b)$ lies on the graph of $y = f(x) = \sqrt{x}$, it must be so that $b = f(a) = \sqrt{a}$. This is our relationship. Now we can substitute $(\sqrt{b})$ in for $b$ in our area equation to find area completely in terms of $a$:

$$A(a) = \frac{1}{2}a(\sqrt{a}) = \frac{1}{2}a\sqrt{a}.$$

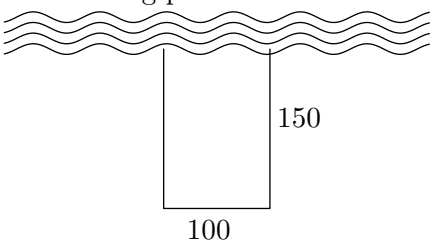
EXAMPLE 3.2.5. You own land along side a river, which you intend to fence. You have 400 feet of fence which you intend to fence three sides of a rectangular plot, allowing the river to compose the final side.



**(i)** If the side parallel to the river is to be 100 feet, find the area of the resulting plot of land.

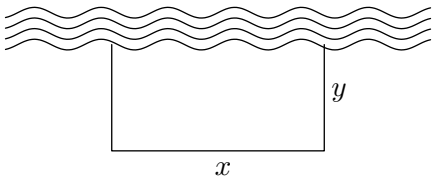
SOLUTION: If we use all of the fencing, then we would use 100 feet along the side parallel to the river. Of the remaining 300 feet $(400 - 100)$ we can split it in half and put 150 feet $(\frac{400-100}{2})$ on each side. This would make the enclosed area be

$$A = 100 \times 150 = 15000$$



**(ii)** Let $x$ be the length of fence parallel to the river. Find the function $A(x)$ which computes the area of the enclosed rectangular area.

Solution: To find the area of this rectangular region, we start with the equation for the area of a rectangle. Let $x$ and $y$ be the dimensions of the rectangle as depicted. Then




$$A = x \cdot y.$$

To make $A$ be a function of $x$, we need to be able to compute $y$ in terms of $x$. Notice in the example before, to figure out the value of $y$, we first figured out how the fence was being allocated. There are two lengths of $y$ and a length of $x$ which would be needing fence, and if we use all of the fence, this means that

$$x + 2 \cdot y = 400$$

If we solve for $y$, we find that

$$\begin{aligned} 2y &= 400 - x \\ y &= \tfrac{1}{2}(400 - x) \end{aligned}$$

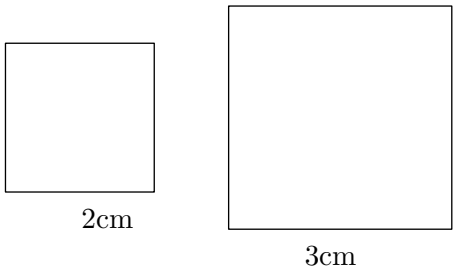
Using this computed value for $y$ in terms of $x$, we can substitute in the equation for area to get

$$A = A(x) = x \cdot \left(\frac{1}{2}(400 - x)\right) = x \cdot (200 - \frac{1}{2}x) = 200x - \frac{1}{2}x^2.$$

Example 3.2.6. A piece of wire which is 20cm long is to be cut in to two smaller wires.

**(i)** If one length is 8cm (and thus the other is 12cm) and each is bent into the shape of a square, what would be the enclosed area?

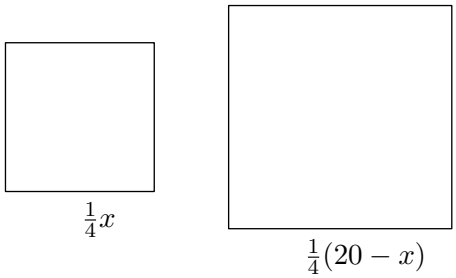
Solution: One needs to recognize that the cut length of wire becomes the perimeter of the resulting shape. If the perimeter of the first square is 8cm, then the side length would be one fourth of 8cm, or 2cm. So the area of the first square would be $2^2 = 4\text{cm}^2$. The second square will have perimeter 12cm, which means its side length would be 3cm, so its area would be $3^2 = 9\text{cm}^2$. So the total area enclosed in both squares would be $9 + 4 = 13\text{cm}^2$.




**(ii)** Let $x$ be one of the cut wire lengths. Find the function $A(x)$ which computes the enclosed area if each length of wire is shaped into a square.

SOLUTION: If one of our cut lengths is $x$, then the other must be $20 - x$. If the piece of length $x$ is shaped into a square, that means the square's perimeter is $x$. So its side length is $\frac{x}{4}$. Thus its enclosed area would be

$$\left(\frac{x}{4}\right)^2 = \frac{x^2}{16}.$$

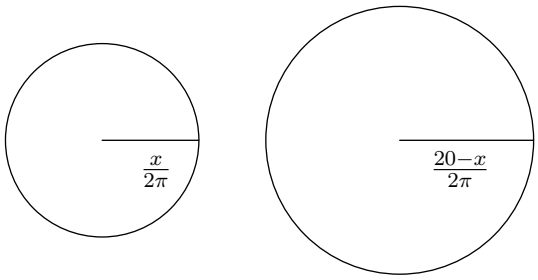



The other piece is of length $20 - x$, meaning the square into which it is shaped will have a side length of $\frac{20-x}{4} = 5 - \frac{x}{4}$. So the enclosed area of this square would be $\left(5 - \frac{1}{4}x\right)^2$. This means that the total enclosed area by both squares ($A(x)$) would be

$$A(x) = \frac{x^2}{16} + \left(5 - \frac{1}{4}x\right)^2.$$

**(iii)** Let $x$ be one of the cut wire lengths. Find the function $B(x)$ which computes the enclosed area if each length of wire is shaped into a circle.

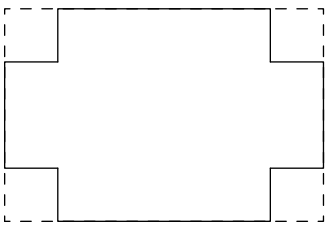
SOLUTION: Here, if the wire of length $x$ is shaped into a circle, then its *circumference* is $x$. So its radius for the first circle can be determined by $2\pi r = x$, so $r = \dfrac{x}{2\pi}$. This means the area of this circle would be $\pi\left(\dfrac{x}{2\pi}\right)^2$. Similarly, the area of the second circle is computed to be $\pi\left(\dfrac{20-x}{2\pi}\right)^2$. So the area function $B(x)$ would be

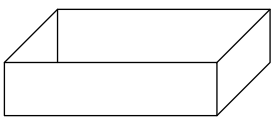



$$B(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \pi\left(\frac{20-x}{2\pi}\right)^2 = \frac{x^2 + (20-x)^2}{4\pi}.$$

(Try to simplify to this!)

EXAMPLE 3.2.7. An open-top box is created by taking a $8 \times 12$ in$^2$ piece of cardboard and cutting squares from the corners, then folding up the sides.

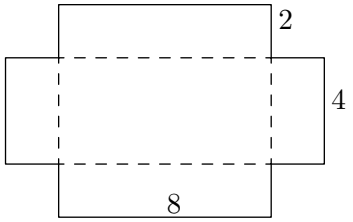


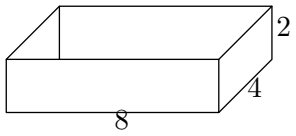


**(i)** If you cut out $2 \times 2$ squares from each corner, what would be the volume of the resulting box?

Solution: If we cut out squares of side length 2, then the lengths which become the length and width of the base become $8 - 2(2) = 4$in and $12 - 2(2) = 8$in. The resulting height of the box is the same as the cut length, 2in. So the volume of this box would be

$$\text{Volume } = 2 \times 4 \times 8 = 64\text{in}^3.$$

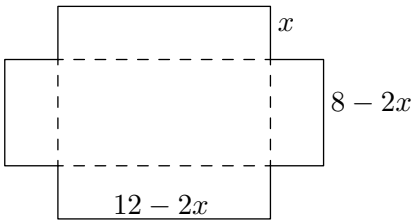


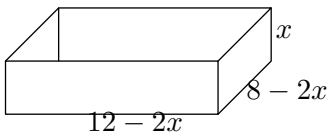




**(ii)** Let $x$ be the length of the side of a cut-out square. Find the function $V(x)$ which computes the resulting volume of the box.

Solution: To compute the volume of this box, we need to know the length, width, and height in terms of $x$. As noted above, the height would be precisely $x$. The way we computed our other dimensions above is to subtract twice $x$ from each of the original dimensions, so our length would be $12 - 2x$ and the width would be $8 - 2x$. Therefore, the volume of this box would be

$$V(x) = x(12 - 2x)(8 - 2x).$$




### 3.2.1 Exercises

### 3.2.2 Answers

## 3.3 QUADRATIC FUNCTIONS

You learned about quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

DEFINITION 3.4. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

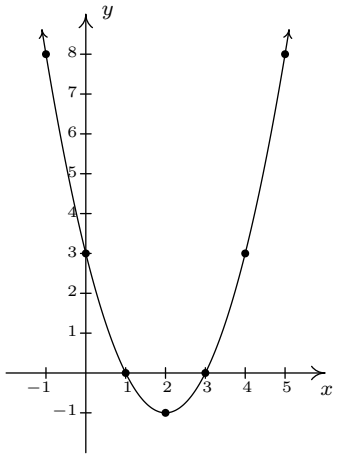
where $a$, $b$, and $c$ are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

EXAMPLE 3.3.1. Graph each of the following quadratic functions. Find the zeros of each function and the $x$- and $y$-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing, or constant and find the relative and absolute extrema, if they exist.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = -2(x-3)^2 + 1$.

SOLUTION.

1. To find the zeros of $f$, we set $f(x) = 0$ and solve the equation $x^2 - 4x + 3 = 0$. Factoring gives us $(x-3)(x-1) = 0$ so that $x = 3$ or $x = 1$. The $x$-intercepts are then $(1, 0)$ and $(3, 0)$. To find the $y$-intercept, we set $x = 0$ and find that $y = f(0) = 3$. Hence, the $y$-intercept is $(0, 3)$. Plotting additional points, we get



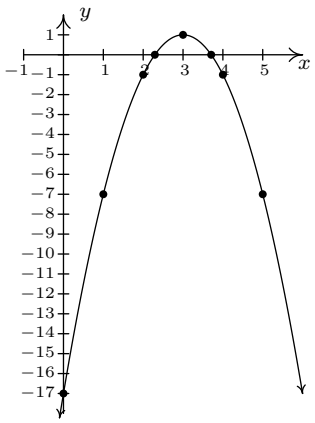

$f(x) = x^2 - 4x + 3$

From the graph, we see the domain is $(-\infty, \infty)$ and the range is $[-1, \infty)$. The function $f$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$. A relative minimum occurs at the point $(2, -1)$ and the value $-1$ is both the relative and absolute minimum of $f$.

2. Note that the formula for $g(x)$ doesn't match the form given in Definition 3.4. However, if we took the time to expand $g(x) = -2(x-3)^2 + 1$, we would get $g(x) = -2x^2 + 12x - 17$ which does match with Definition 3.4. When we find the zeros of $g$, we can use either formula, since both are equivalent. Using the formula which was given to us, we get

$$\begin{aligned}
g(x) &= 0 && \\
-2(x-3)^2 + 1 &= 0 && \\
-2(x-3)^2 &= -1 && \\
(x-3)^2 &= \frac{1}{2} && \text{divide by } -2 \\
x - 3 &= \pm\sqrt{\frac{1}{2}} && \text{extract square roots} \\
x - 3 &= \pm\frac{\sqrt{2}}{2} && \text{rationalize the denominator} \\
x &= 3 \pm \frac{\sqrt{2}}{2} && \\
x &= \frac{6 \pm \sqrt{2}}{2} && \text{get a common denominator}
\end{aligned}$$

Hence, we have two $x$-intercepts: $\left(\frac{6+\sqrt{2}}{2}, 0\right)$ and $\left(\frac{6-\sqrt{2}}{2}, 0\right)$. (The inquisitive reader may wonder what we would have done had we chosen to set the expanded form of $g(x)$ equal to zero. Since $-2x^2 + 12x - 17$ does not factor nicely, we would have had to resort to other methods, which are reviewed later in this section, to solve $-2x^2 + 12x - 17 = 0$.) To find the $y$-intercept, we set $x = 0$ and get $g(0) = -17$. Our $y$-intercept is then $(0, -17)$. Plotting some additional points, we get




$g(x) = -2(x-3)^2 + 1$

The domain of $g$ is $(-\infty, \infty)$ and the range is $(-\infty, 1]$. The function $g$ is increasing on $(-\infty, 3]$ and decreasing on $[3, \infty)$. The relative maximum occurs at the point $(3, 1)$ with 1 being both the relative and absolute maximum value of $g$. □

Hopefully the previous examples have reminded you of some of the basic characteristics of the graphs of quadratic equations. First and foremost, the graph of $y = ax^2 + bx + c$ where $a$, $b$, and $c$ are real numbers with $a \neq 0$ is called a **parabola**. If the coefficient of $x^2$, $a$, is positive, the parabola opens upwards; if $a$ is negative, it opens downwards, as illustrated below.[1]

Graphs of $y = ax^2 + bx + c$.

The point at which the relative minimum (if $a > 0$) or relative maximum (if $a < 0$) occurs is called the **vertex** of the parabola. Note that each of the parabolas above is symmetric about the dashed vertical line which contains its vertex. This line is called the **axis of symmetry** of the parabola. As you may recall, there are two ways to quickly find the vertex of a parabola, depending on which form we are given. The results are summarized below.

[1] We will justify the role of $a$ in the behavior of the parabola later in the section.

Equation 3.4. **Vertex Formulas for Quadratic Functions**: Suppose $a$, $b$, $c$, $h$, and $k$ are real numbers with $a \neq 0$.

- If $f(x) = a(x-h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point $(h, k)$.
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Example 3.3.2. Use Equation 3.4 to find the vertex of the graphs in Example 3.3.1.

Solution.

1. The formula $f(x) = x^2 - 4x + 3$ is in the form $f(x) = ax^2 + bx + c$. We identify $a = 1$, $b = -4$, and $c = 3$, so that
$$-\frac{b}{2a} = -\frac{-4}{2(1)} = 2,$$
and
$$f\left(-\frac{b}{2a}\right) = f(2) = -1,$$
so the vertex is $(2, -1)$ as previously stated.
2. We see that the formula $g(x) = -2(x-3)^2 + 1$ is in the form $g(x) = a(x-h)^2 + k$. We identify $a = -2$, $x - h$ as $x - 3$ (so $h = 3$), and $k = 1$ and get the vertex $(3, 1)$, as required. □

The formula $f(x) = a(x-h)^2 + k$, $a \neq 0$ in Equation 3.4 is sometimes called the **standard form** of a quadratic function; the formula $f(x) = ax^2 + bx + c$, $a \neq 0$ is sometimes called the **general form** of a quadratic function.

To see why the formulas in Equation 3.4 produce the vertex, let us first consider a quadratic function in standard form. If we consider the graph of the equation $y = a(x-h)^2 + k$ we see that when $x = h$, we get $y = k$, so $(h, k)$ is on the graph. If $x \neq h$, then $x - h \neq 0$ and so $(x-h)^2$ is a positive number. If $a > 0$, then $a(x-h)^2$ is positive, and so $y = a(x-h)^2 + k$ is always a number larger than $k$. That means that when $a > 0$, $(h, k)$ is the lowest point on the graph and thus the parabola must open upwards, making $(h, k)$ the vertex. A similar argument shows that if $a < 0$, $(h, k)$ is the highest point on the graph, so the parabola opens downwards, and $(h, k)$ is also the vertex in this case. Alternatively, we can apply the machinery in Section 2.5. The vertex of the parabola $y = x^2$ is easily seen to be the origin, $(0, 0)$. We leave it to the reader to convince oneself that if we apply any of the transformations in Section 2.5 (shifts, reflections, and/or scalings) to $y = x^2$, the vertex of the resulting parabola will always be the point the graph corresponding to $(0, 0)$. To obtain the formula $f(x) = a(x-h)^2 + k$, we start with $g(x) = x^2$ and first define $g_1(x) = ag(x) = ax^2$. This is results in a vertical scaling and/or reflection.[2] Since

[2] Just a scaling if $a > 0$. If $a < 0$, there is a reflection involved.

we multiply the output by $a$, we multiply the $y$-coordinates on the graph of $g$ by $a$, so the point $(0,0)$ remains $(0,0)$ and remains the vertex. Next, we define $g_2(x) = g_1(x-h) = a(x-h)^2$. This induces a horizontal shift right or left $h$ units[3] moves the vertex, in either case, to $(h,0)$. Finally, $f(x) = g_2(x) + k = a(x-h)^2 + k$ which effects a vertical shift up or down $k$ units[4] resulting in the vertex moving from $(h,0)$ to $(h,k)$.

To verify the vertex formula for a quadratic function in general form, we complete the square to convert the general form into the standard form.[5]

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{complete the square} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{factor; get a common denominator} \end{aligned}$$

Comparing this last expression with the standard form, we identify $(x-h)$ as $\left(x + \frac{b}{2a}\right)$ so that $h = -\frac{b}{2a}$. Instead of memorizing the value $k = \frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$. As such, we have derived the vertex formula for the general form as well. Note that the value $a$ plays the exact same role in both the standard and general equations of a quadratic function – it is the coefficient of $x^2$ in each. No matter what the form, if $a > 0$, the parabola opens upwards; if $a < 0$, the parabola opens downwards.

Now that we have the completed square form of the general form of a quadratic function, it is time to remind ourselves of the **quadratic formula**. In a function context, it gives us a means to find the zeros of a quadratic function in general form.

EQUATION 3.5. **The Quadratic Formula:** If $a$, $b$, $c$ are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

[3] Right if $h > 0$, left if $h < 0$.

[4] Up if $k > 0$, down if $k < 0$

[5] Actually, we could also take the standard form, $f(x) = a(x-h)^2 + k$, expand it, and compare the coefficients of it and the general form to deduce the result. However, we will have another use for the completed square form of the general form of a quadratic, so we'll proceed with the conversion.

Assuming the conditions of Equation 3.5, the solutions to $ax^2 + bx + c = 0$ are precisely the zeros of $f(x) = ax^2 + bx + c$. We have shown an equivalent formula for $f$ is

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Hence, an equation equivalent to $ax^2 + bx + c = 0$ is

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Solving gives

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac - b^2}{4a} \\
\frac{1}{a}\left[a\left(x + \frac{b}{2a}\right)^2\right] &= \frac{1}{a}\left(\frac{b^2 - 4ac}{4a}\right) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{extract square roots} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in

determining the nature of the solutions to a quadratic equation. It is given a special name and is discussed below.

DEFINITION 3.5. If $a$, $b$, $c$ are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

THEOREM 3.1. **Discriminant Trichotomy:** Let $a$, $b$, and $c$ be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

The proof of Theorem 3.1 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

EXAMPLE 3.3.3. The **profit** function for a product is defined by the equation Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. Recall from Example 3.1.7 that the weekly revenue, in dollars, made by selling $x$ PortaBoy Game Systems is given by $R(x) = -1.5x^2 + 250x$. The cost, in dollars, to produce $x$ PortaBoy Game Systems is given in Example 3.1.5 as $C(x) = 80x + 150$, $x \geq 0$.

1. Determine the weekly profit function, $P(x)$.
2. Graph $y = P(x)$. Include the $x$- and $y$-intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of $P$.
4. Interpret the vertex of the graph of $y = P(x)$.
5. Recall the weekly price-demand equation for PortaBoys is: $p(x) = -1.5x + 250$, where $p(x)$ is the price per PortaBoy, in dollars, and $x$ is the weekly sales. What should the price per system be in order to maximize profit?

SOLUTION.

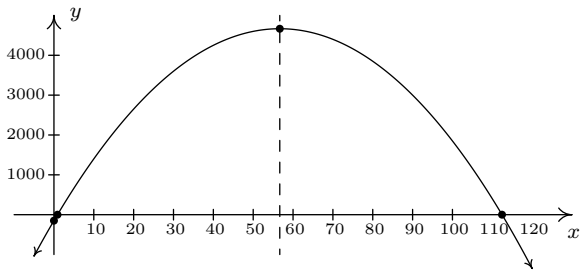
1. To find the profit function $P(x)$, we subtract

$$P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150.$$

2. To find the $x$-intercepts, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 3.5. Identifying $a = -1.5$, $b = 170$, and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \end{aligned}$$

We get two $x$-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the $y$-intercept, we set $x = 0$ and find $y = P(0) = -150$ for a $y$-intercept of $(0, -150)$. To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ is in the general form of a quadratic function and appeal to Equation 3.4. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$. To find the $y$-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ and find our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$. To sketch a reasonable graph, we approximate the $x$-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. (Note that in order to get the $x$-intercepts and the vertex to show up in the same picture, we had to scale the $x$-axis differently than the $y$-axis. This results in the left-hand $x$-intercept and the $y$-intercept being uncomfortably close to each other and to the origin in the picture.)




3. The zeros of $P$ are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. Since $P$ represents the weekly profit, $P(x) = 0$ means the weekly profit is \$0. Sometimes, these values of $x$ are called the 'break-even' points of the profit function, since these are places where the revenue equals the cost; in other words we gave sold enough

product to recover the cost spent to make the product. More importantly, we see from the graph that as long as $x$ is between 0.89 and 112.44, the graph $y = P(x)$ is above the $x$-axis, meaning $y = P(x) > 0$ there. This means that for these values of $x$, a profit is being made. Since $x$ represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.

4. From the graph, we see the maximum value of $P$ occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, $x$ represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude we will make a maximum profit of \$4666.50 if we sell 57 game systems.

5. In the previous part, we found we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. The price should be set at \$164.50. □

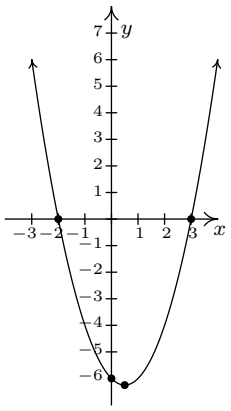
We conclude this section with a more complicated absolute value function.

EXAMPLE 3.3.4. Graph $f(x) = |x^2 - x - 6|$.

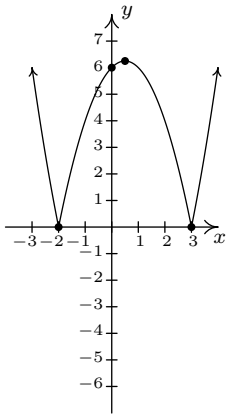
SOLUTION. Using the definition of absolute value, Definition 0.2, we have

$$f(x) = \begin{cases} -\left(x^2 - x - 6\right), & \text{if} \quad x^2 - x - 6 < 0 \\ x^2 - x - 6, & \text{if} \quad x^2 - x - 6 \geq 0 \end{cases}$$

The trouble is that we have yet to develop any analytic techniques to solve nonlinear inequalities such as $x^2 - x - 6 < 0$. You won't have to wait long; this is one of the main topics of Section 3.4. Nevertheless, we can attack this problem graphically. To that end, we graph $y = g(x) = x^2 - x - 6$ using the intercepts and the vertex. To find the $x$-intercepts, we solve $x^2 - x - 6 = 0$. Factoring gives $(x - 3)(x + 2) = 0$ so $x = -2$ or $x = 3$. Hence, $(-2, 0)$ and $(3, 0)$ are $x$-intercepts. The $y$-intercept is found by setting $x = 0$, $(0, -6)$. To find the vertex, we find $x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}$, and $y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = -\frac{25}{4} = -6.25$. Plotting, we get the parabola seen below on the left. To obtain points on the graph of $y = f(x) = |x^2 - x - 6|$, we can take points on the graph of $g(x) = x^2 - x - 6$ and apply the absolute value to each of the $y$ values on the parabola. We see from the graph of $g$ that for $x \leq -2$ or $x \geq 3$, the $y$ values on the parabola are greater than or equal to zero (since the graph is on or above the $x$-axis), so the absolute value leaves this portion of the graph alone. For $x$ between $-2$ and 3, however, the $y$ values on the parabola are negative. For example, the point $(0, -6)$ on $y = x^2 - x - 6$ would result in the point $(0, |-6|) = (0, -(-6)) = (0, 6)$ on the graph of $f(x) = |x^2 - x - 6|$. Proceeding in this manner for all points with $x$-coordinates between $-2$ and 3 results in the graph seen above on the right.




$y = g(x) = x^2 - x - 6$




$y = f(x) = |x^2 - x - 6|$

□

If we take a step back and look at the graphs of $g$ and $f$ in the last example, we notice that to obtain the graph of $f$ from the graph of $g$, we reflect a *portion* of the graph of $g$ about the $x$-axis. We can see this analytically by substituting $g(x) = x^2 - x - 6$ into the formula for $f(x)$ and calling to mind Theorem 2.4 from Section 2.5.

$$f(x) = \begin{cases} -g(x), & \text{if} \quad g(x) < 0 \\ g(x), & \text{if} \quad g(x) \geq 0 \end{cases}$$

The function $f$ is defined so that when $g(x)$ is negative (i.e., when its graph is below the $x$-axis), the graph of $f$ is its refection across the $x$-axis. This is a general template to graph functions of the form $f(x) = |g(x)|$. From this perspective, the graph of $f(x) = |x|$ can be obtained by reflection the portion of the line $g(x) = x$ which is below the $x$-axis back above the $x$-axis creating the characteristic '$\vee$' shape.

### 3.3.1 EXERCISES

1. Graph each of the following quadratic functions. Find the $x$- and $y$-intercepts of each graph, if any exist. If it is given in the general form, convert it into standard form. Find the domain and range of each function and list the intervals on which the function is increasing or decreasing. Identify the vertex and the axis of symmetry and determine whether the vertex yields a relative and absolute maximum or minimum.

   (a) $f(x) = x^2 + 2$
   (b) $f(x) = -(x+2)^2$
   (c) $f(x) = x^2 - 2x - 8$
   (d) $f(x) = -2(x+1)^2 + 4$
   (e) $f(x) = 2x^2 - 4x - 1$
   (f) $f(x) = -3x^2 + 4x - 7$
   (g) $f(x) = -3x^2 + 5x + 4$
   (h) [6] $f(x) = x^2 - \frac{1}{100}x - 1$

2. Graph $f(x) = |1 - x^2|$

3. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$

4. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the $x$-coordinate of the vertex is the midpoint of the zeros.

5. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time $t$ of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where $h$ is in meters, $t$ is in seconds, $v_0$ is the object's initial velocity in meters per second and $s_0$ is its initial position in meters.

   (a) What is the applied domain of this function?
   (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.
   (c) Come up with a scenario in which $s_0 < 0$.
   (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?
   (e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?
   (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

6. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking $x$ cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$. How many cookies should they bake in order to maximize their profit?

[6] We have already seen the graph of this function. It was used as an example in Section 2.4 to show how the graphing calculator can be misleading.

7. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[7] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

8. What is the largest rectangular area one can enclose with 14 inches of string?

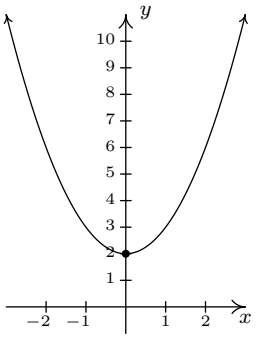
9. Solve the following quadratic equations for the indicated variable.

   (a) $x^2 - 10y^2 = 0$ for $x$
   (b) $y^2 - 4y = x^2 - 4$ for $x$
   (c) $x^2 - mx = 1$ for $x$
   (d) $-gt^2 + v_0 t + s_0 = 0$ for $t$ (Assume $g \neq 0$.)
   (e) $y^2 - 3y = 4x$ for $y$
   (f) $y^2 - 4y = x^2 - 4$ for $y$

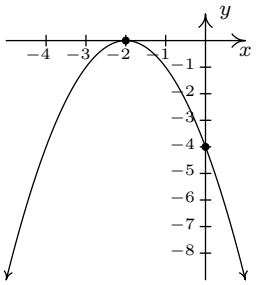
[7] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise **??** in Section 7.5 what shape a free hanging cable makes.

### 3.3.2 ANSWERS

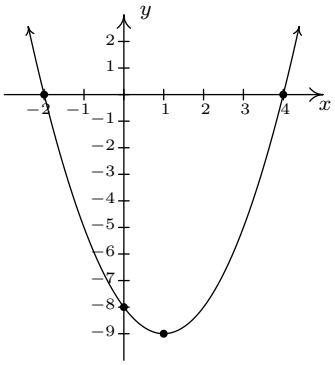
1. (a) $f(x) = x^2 + 2$
   No $x$-intercepts
   $y$-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



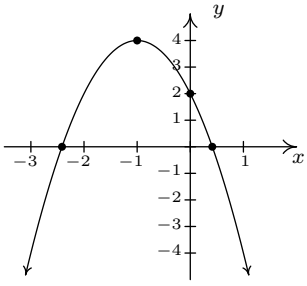

   (b) $f(x) = -(x + 2)^2$
   $x$-intercept $(-2, 0)$
   $y$-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$



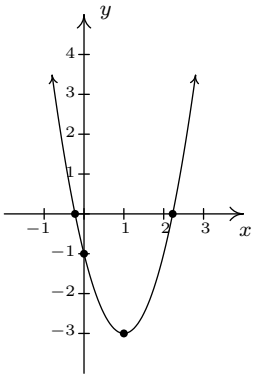

   (c) $f(x) = x^2 - 2x - 8 = (x - 1)^2 - 9$
   $x$-intercepts $(-2, 0)$ and $(4, 0)$
   $y$-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$



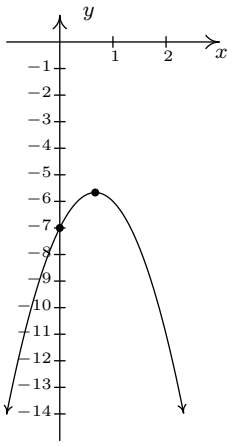

   (d) $f(x) = -2(x + 1)^2 + 4$
   $x$-intercepts $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$
   $y$-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 4])$
   Increasing on $(-\infty, -1]$
   Decreasing on $[-1, \infty)$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $x = -1$



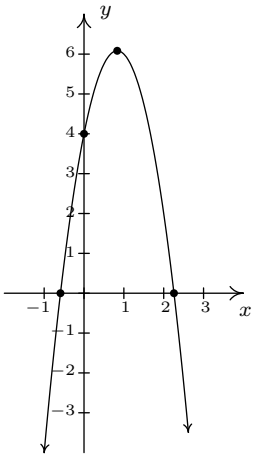
(e) $f(x) = 2x^2 - 4x - 1 = 2(x-1)^2 - 3$
$x$-intercepts $\left(-1 - \frac{\sqrt{6}}{2}, 0\right)$ and $\left(-1 + \frac{\sqrt{6}}{2}, 0\right)$
$y$-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$
Increasing on $[1, \infty)$
Decreasing on $(-\infty, 1]$
Vertex $(1, -3)$ is a minimum
Axis of symmetry $x = 1$



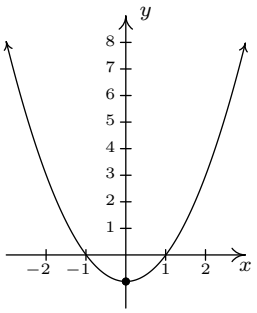

(f) $f(x) = -3x^2 + 4x - 7 = -3(x - \frac{2}{3})^2 - \frac{17}{3}$
No $x$-intercepts
$y$-intercept $(0, -7)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, -\frac{17}{3}]$
Increasing on $(-\infty, \frac{2}{3}]$
Decreasing on $[\frac{2}{3}, \infty)$
Vertex $(\frac{2}{3}, -\frac{17}{3})$ is a maximum
Axis of symmetry $x = \frac{2}{3}$



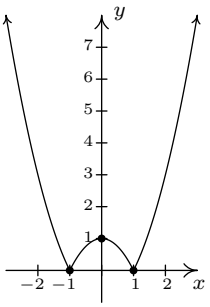

(g) $f(x) = -3x^2 + 5x + 4 = -3\left(x - \frac{5}{6}\right)^2 + \frac{73}{12}$
$x$-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
$y$-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a minimum
Axis of symmetry $x = \frac{5}{6}$



(h) $f(x) = x^2 - \frac{1}{100}x - 1 = \left(x - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
$x$-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
$y$-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[8]
Axis of symmetry $x = \frac{1}{200}$




2. $y = |1 - x^2|$




3. $\left(\frac{3-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right), \left(\frac{3+\sqrt{7}}{2}, \frac{-1-\sqrt{7}}{2}\right)$

5. (a) The applied domain is $[0, \infty)$.

(d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.

(e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.

(f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

6. 495 cookies

7. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

8. The largest rectangle has area $12.25\text{in}^2$.

[8] You'll need to use your calculator to zoom in far enough to see that the vertex is not the $y$-intercept.

9. (a) $x = \pm y\sqrt{10}$

   (b) $x = \pm(y - 2)$

   (c) $x = \dfrac{m \pm \sqrt{m^2 + 4}}{2}$

   (d) $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

   (e) $y = \dfrac{3 \pm \sqrt{16x + 9}}{2}$

   (f) $y = 2 \pm x$

## 3.4 INEQUALITIES

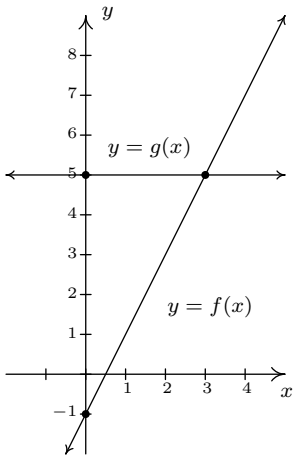
In this section, not only do we develop techniques for solving various classes of inequalities analytically, we also look at them graphically. The next example motivates the core ideas.

EXAMPLE 3.4.1. Let $f(x) = 2x - 1$ and $g(x) = 5$.

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) > g(x)$.
4. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

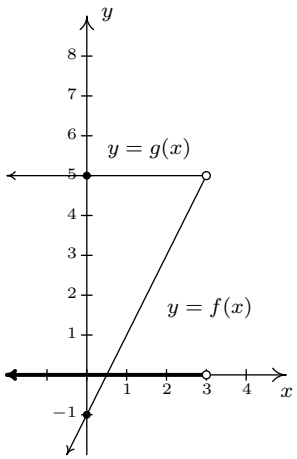
SOLUTION.

1. To solve $f(x) = g(x)$, we replace $f(x)$ with $2x - 1$ and $g(x)$ with 5 to get $2x - 1 = 5$. Solving for $x$, we get $x = 3$.
2. The inequality $f(x) < g(x)$ is equivalent to $2x - 1 < 5$. Solving gives $x < 3$ or $(-\infty, 3)$.
3. To find where $f(x) > g(x)$, we solve $2x - 1 > 5$. We get $x > 3$, or $(3, \infty)$.
4. To graph $y = f(x)$, we graph $y = 2x - 1$, which is a line with a $y$-intercept of $(0, -1)$ and a slope of 2. The graph of $y = g(x)$ is $y = 5$ which is a horizontal line through $(0, 5)$.

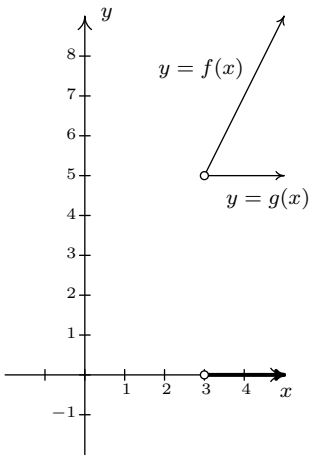



To see the connection between the graph and the algebra, we recall the Fundamental Graphing Principle for Functions in Section 2.4: the point $(a, b)$ is on the graph of $f$ if and only if $f(a) = b$. In other words, a generic point on the graph of $y = f(x)$ is $(x, f(x))$, and a generic

point on the graph of $y = g(x)$ is $(x, g(x))$. When we seek solutions to $f(x) = g(x)$, we are looking for values $x$ whose $y$ values on the graphs of $f$ and $g$ are the same. In part 1, we found $x = 3$ is the solution to $f(x) = g(x)$. Sure enough, $f(3) = 5$ and $g(3) = 5$ so that the point $(3, 5)$ is on both graphs. We say the graphs of $f$ and $g$ **intersect** at $(3, 5)$. In part 2, we set $f(x) < g(x)$ and solved to find $x < 3$. For $x < 3$, the point $(x, f(x))$ is **below** $(x, g(x))$ since the $y$ values on the graph of $f$ are less than the $y$ values on the graph of $g$ there. Analogously, in part 3, we solved $f(x) > g(x)$ and found $x > 3$. For $x > 3$, note that the graph of $f$ is **above** the graph of $g$, since the $y$ values on the graph of $f$ are greater than the $y$ values on the graph of $g$ for those values of $x$.




$f(x) < g(x)$




$f(x) > g(x)$

□

The preceding example demonstrates the following, which is a consequence of the Fundamental Graphing Principle for Functions.

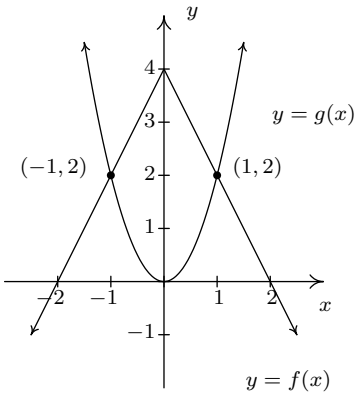
**Graphical Interpretation of Equations and Inequalities**

Suppose $f$ and $g$ are functions.

- The solutions to $f(x) = g(x)$ are precisely the $x$ values where the graphs of $y = f(x)$ and $y = g(x)$ intersect.
- The solutions to $f(x) < g(x)$ are precisely the $x$ values where the graph of $y = f(x)$ is **below** the graph of $y = g(x)$.
- The solutions to $f(x) > g(x)$ are precisely the $x$ values where the graph of $y = f(x)$ **above** the graph of $y = g(x)$.

The next example turns the tables and furnishes the graphs of two functions and asks for solutions to equations and inequalities.

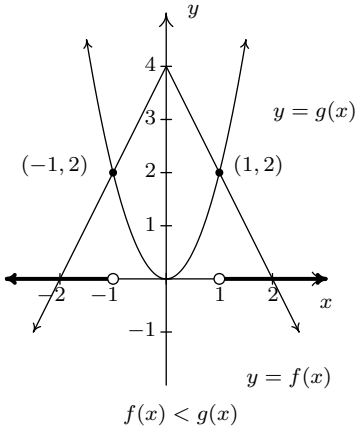
EXAMPLE 3.4.2. The graphs of $f$ and $g$ are below. The graph of $y = f(x)$ resembles the upside down $\vee$ shape of an absolute value function while the graph of $y = g(x)$ resembles a parabola. Use these graphs to answer the following questions.



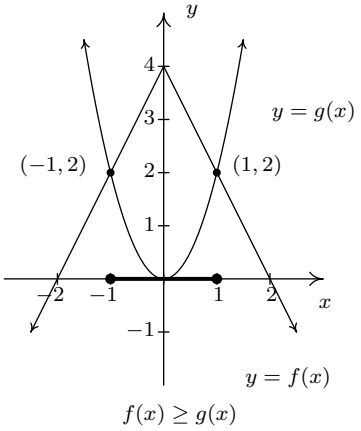

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) \geq g(x)$.

SOLUTION.

1. To solve $f(x) = g(x)$, we look for where the graphs of $f$ and $g$ intersect. These appear to be at the points $(-1, 2)$ and $(1, 2)$, so our solutions to $f(x) = g(x)$ are $x = -1$ and $x = 1$.
2. To solve $f(x) < g(x)$, we look for where the graph of $f$ is below the graph of $g$. This appears to happen for the $x$ values less than $-1$ and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.
3. To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. We solved the former equation and found $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of $f$ is above the graph of $g$. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.




$f(x) < g(x)$




$f(x) \geq g(x)$

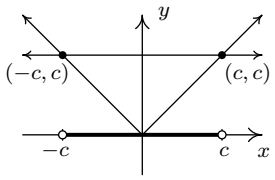
□

We now turn our attention to solving inequalities involving the absolute value. We have the following theorem from Intermediate Algebra to help us.

THEOREM 3.2. **Inequalities Involving the Absolute Value:** Let $c$ be a real number.

- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c \leq 0$, $|x| < c$ has no solution.
- For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- For $c < 0$, $|x| > c$ is true for all real numbers.

We could argue Theorem 3.2 using cases. However, in light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the $x$-axis through $(0, c)$. To solve $|x| < c$, we are looking for the $x$ values where the graph of $y = |x|$ is below the graph of $y = c$. We know the graphs intersect when $|x| = c$, and this happens when $x = c$ or $x = -c$. Graphing, we get



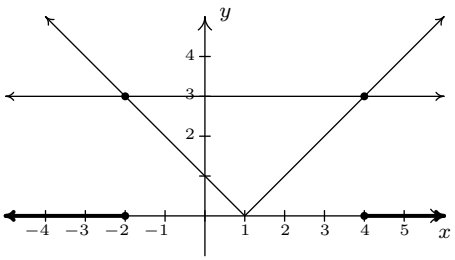

We see the graph of $y = |x|$ is below $y = c$ for $x$ between $-c$ and $c$, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in Theorem 3.2 can be shown similarly.

EXAMPLE 3.4.3. Solve the following inequalities analytically; check your answers graphically.

1. $|x - 1| \geq 3$
2. $4 - 3|2x + 1| > -2$
3. $2 < |x - 1| \leq 5$
4. $|x + 1| \geq \dfrac{x + 4}{2}$

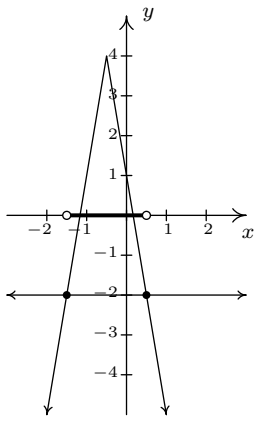
SOLUTION.

1. To solve $|x - 1| \geq 3$, we seek solutions to $|x - 1| > 3$ as well as solutions to $|x - 1| = 3$. From Theorem 3.2, $|x - 1| > 3$ is equivalent to $x - 1 < -3$ or $x - 1 > 3$. From Theorem 0.2, $|x - 1| = 3$ is equivalent to $x - 1 = -3$ or $x - 1 = 3$. Combining these equations with the inequalities, we solve $x - 1 \leq -3$ or $x - 1 \geq 3$. Our answer is $x \leq -2$ or $x \geq 4$, which, in interval notation is $(-\infty, -2] \cup [4, \infty)$. Graphically, we have

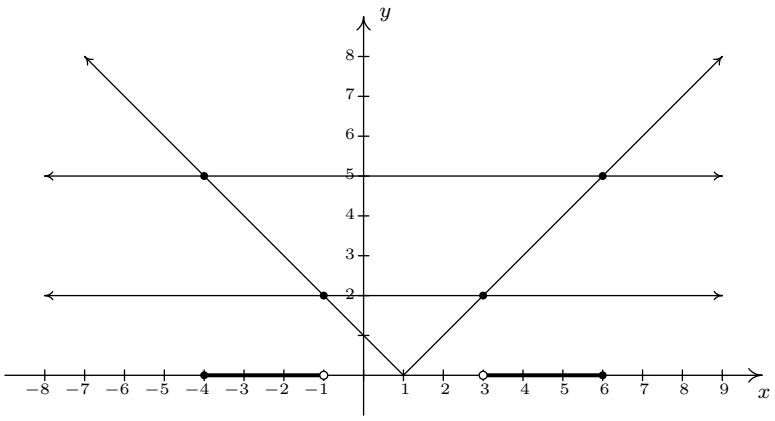



We see the graph of $y = |x - 1|$ (the $\vee$) is above the horizontal line $y = 3$ for $x < -2$ and $x > 4$, and, hence, this is where $|x - 1| > 3$. The two graphs intersect when $x = -2$ and $x = 4$, and so we have graphical confirmation of our analytic solution.

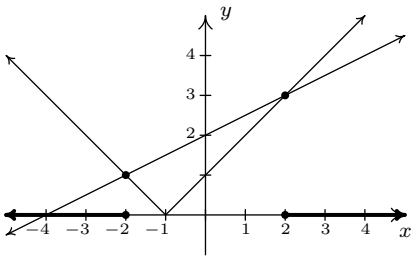
2. To solve $4 - 3|2x + 1| > -2$ analytically, we first isolate the absolute value before applying Theorem 3.2. To that end, we get $-3|2x + 1| > -6$ or $|2x + 1| < 2$. Rewriting, we now have $-2 < 2x + 1 < 2$ so that $-\frac{3}{2} < x < \frac{1}{2}$. In interval notation, we write $\left(-\frac{3}{2}, \frac{1}{2}\right)$. Graphically we see the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for $x$ values between $-\frac{3}{2}$ and $\frac{1}{2}$.




3. Rewriting the compound inequality $2 < |x - 1| \leq 5$ as '$2 < |x - 1|$ and $|x - 1| \leq 5$' allows us to solve each piece using Theorem 3.2. The first inequality, $2 < |x - 1|$ can be re-written as $|x - 1| > 2$ and so $x - 1 < -2$ or $x - 1 > 2$. We get $x < -1$ or $x > 3$. Our solution to the first inequality is then $(-\infty, -1) \cup (3, \infty)$. For $|x - 1| \leq 5$, we combine results in Theorems 0.2 and 3.2 to get $-5 \leq x - 1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of $x$ which satisfy both parts of the inequality, and so we take what's called the 'set theoretic intersection' of $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to obtain $[-4, -1) \cup (3, 6]$. Graphically, we see the graph of $y = |x - 1|$ is 'between' the horizontal lines $y = 2$ and $y = 5$ for $x$ values between $-4$ and $-1$ as well as those between 3 and 6. Including the $x$ values where $y = |x - 1|$ and $y = 5$ intersect, we get

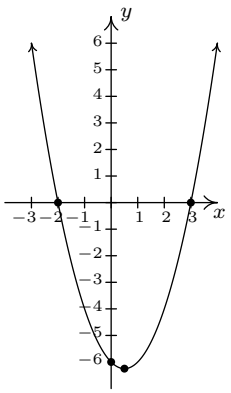



We need to exercise some special caution when solving $|x+1| \geq \frac{x+4}{2}$. When variables are both inside and outside of the absolute value, it's usually best to refer to the definition of absolute value, Definition 0.2, to remove the absolute values and proceed from there. To that end, we have $|x+1| = -(x+1)$ if $x < -1$ and $|x+1| = x+1$ if $x \geq -1$. We break the inequality into cases, the first case being when $x < -1$. For these values of $x$, our inequality becomes $-(x+1) \geq \frac{x+4}{2}$. Solving, we get $-2x-2 \geq x+4$, so that $-3x \geq 6$, which means $x \leq -2$. Since all of these solutions fall into the category $x < -1$, we keep them all. For the second case, we assume $x \geq -1$. Our inequality becomes $x+1 \geq \frac{x+4}{2}$, which gives $2x+2 \geq x+4$ or $x \geq 2$. Since all of these values of $x$ are greater than or equal to $-1$, we accept all of these solutions as well. Our final answer is $(-\infty, -2] \cup [2, \infty)$.



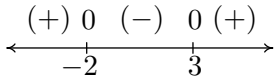

□

We now turn our attention to quadratic inequalities. In the last example of Section 3.3, we needed to determine the solution to $x^2 - x - 6 < 0$. We will now re-visit this problem using some of the techniques developed in this section not only to reinforce our solution in Section 3.3, but to also help formulate a general analytic procedure for solving all quadratic inequalities. If we consider $f(x) = x^2 - x - 6$ and $g(x) = 0$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of $x$ for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = g(x) = 0$ (the $x$-axis.) We've provided the graph again for reference.




$y = x^2 - x - 6$

We can see that the graph of $f$ does dip below the $x$-axis between its two $x$-intercepts. The zeros of $f$ are $x = -2$ and $x = 3$ in this case and they divide the domain (the $x$-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of $f$ is above the $x$-axis; in other words, $f(x) > 0$ for all $x$ in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all $x$ in $(-2, 3)$, and $f(x) > 0$ for all $x$ in $(3, \infty)$. We can schematically represent this with the **sign diagram** below.




Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of $x$; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of $f$, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2, 3)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. An important property[1] of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the $x$-axis at one point and below the $x$-axis at another point without crossing the $x$-axis. This allows us to determine the sign of **all** of the function values on a given interval by testing the function at just **one** value in the interval. This gives us the following.

[1] We will give this property a name in Chapter 4 and revisit this concept then.

**Steps for Solving a Quadratic Inequality**

1. Rewrite the inequality, if necessary, as a quadratic function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of $f$ and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the intervals which correspond to the correct sign to solve the inequality.

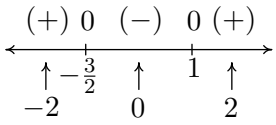
Example 3.4.4. Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

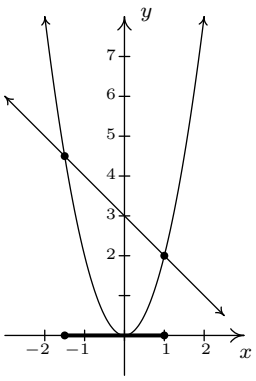
1. $2x^2 \leq 3 - x$
2. $x^2 > 2x + 1$
3. $9x^2 + 4 \leq 12x$
4. $2x - x^2 \geq |x - 1| - 1$

Solution.

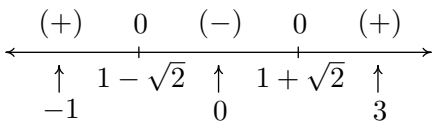
1. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for $x$. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$,or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$, and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[2] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$ (so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$ (so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$); and, $f(2) = 7$ (which means $(+)$ is placed above $(1, \infty)$). Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $\left(-\frac{3}{2}, 1\right)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so or final answer is $\left[-\frac{3}{2}, 1\right]$. To check our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the $x$ values where the graph of $g$ is below that of $h$ (the solution to $g(x) < h(x)$) as well as the two graphs intersect (the solutions to $g(x) = h(x)$.) The graphs of $g$ and $h$ are given on the right with the sign chart on the left.

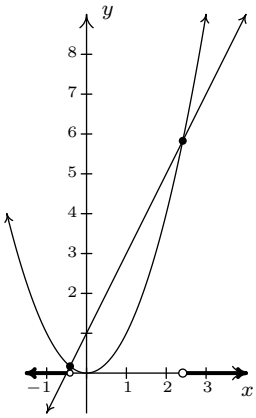
[2] We have to choose something in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.



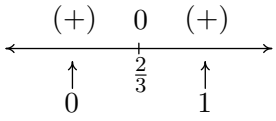

2. Once again, we re-write $x^2 > 2x + 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of $f$, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$, and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$. To check the inequality $x^2 > 2x + 1$ graphically, we set $g(x) = x^2$ and $h(x) = 2x + 1$. We are looking for the $x$ values where the graph of $g$ is above the graph of $h$. As before we present the graphs on the right and the sign chart on the left.

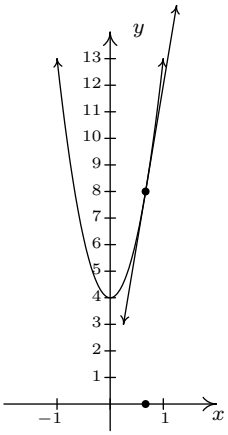


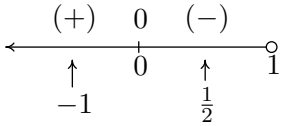

3. To solve $9x^2 + 4 \leq 12x$, as before, we solve $9x^2 - 12x + 4 \leq 0$. Setting $f(x) = 9x^2 - 12x + 4 = 0$, we find the only one zero of $f$, $x = \frac{2}{3}$. This one $x$ value divides the number line into two intervals, from which we choose $x = 0$ and $x = 1$ as test values. We find $f(0) = 4 > 0$ and $f(1) = 1 > 0$. Since we are looking for solutions to $9x^2 - 12x + 4 \leq 0$, we are looking for

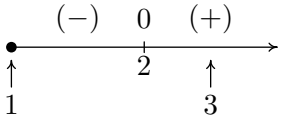
$x$ values where $9x^2 - 12x + 4 < 0$ as well as where $9x^2 - 12x + 4 = 0$. Looking at our sign diagram, there are no places where $9x^2 - 12x + 4 < 0$ (there are no $(-)$), so our solution is only $x = \frac{2}{3}$ (where $9x^2 - 12x + 4 = 0$). We write this as $\left\{\frac{2}{3}\right\}$. Graphically, we solve $9x^2 + 4 \leq 12x$ by graphing $g(x) = 9x^2 + 4$ and $h(x) = 12x$. We are looking for the $x$ values where the graph of $g$ is below the graph of $h$ (for $9x^2 + 4 < 12x$) and where the two graphs intersect ($9x^2 + 4 = 12x$). We see the line and the parabola touch at $\left(\frac{2}{3}, 8\right)$, but the parabola is always above the line otherwise.[3]




4. To solve our last inequality, $2x - x^2 \geq |x - 1| - 1$, we re-write the absolute value using cases. For $x < 1$, $|x - 1| = -(x - 1) = 1 - x$, so we get $2x - x^2 \geq 1 - x - 1$, or $x^2 - 3x \leq 0$. Finding the zeros of $f(x) = x^2 - 3x$, we get $x = 0$ and $x = 3$. However, we are only concerned with the portion of the number line where $x < 1$, so the only zero that we concern ourselves with is $x = 0$. This divides the interval $x < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $x = -1$ and $x = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Solving $x^2 - 3x \leq 0$ for $x < 1$ gives us $[0, 1)$. Next, we turn our attention to the case $x \geq 1$. Here, $|x - 1| = x - 1$, so our original inequality becomes $2x - x^2 \geq x - 1 - 1$, or $x^2 - x - 2 \leq 0$. Setting $g(x) = x^2 - x - 2$, we find the zeros of $g$ to be $x = -1$ and $x = 2$. Of these, only $x = 2$ lies in the region $x \geq 1$, so we ignore $x = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $x = 1$ and $x = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. To solve $g(x) \leq 0$, we have $[1, 2)$.

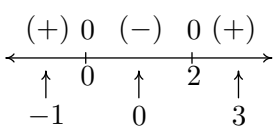


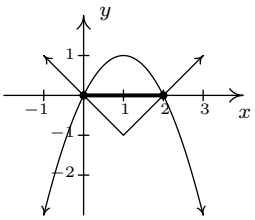




Combining these into one sign diagram, we get our solution is $[0, 2]$. Graphically, to check $2x - x^2 \geq |x - 1| - 1$, we set $h(x) = 2x - x^2$ and $i(x) = |x - 1| - 1$ and look for the $x$ values

[3] In this case, we say the line $y = 12x$ is **tangent** to $y = 9x^2 + 4$ at $\left(\frac{2}{3}, 8\right)$. Finding tangent lines to arbitrary functions is the stuff of legends, I mean, Calculus.

where the graph of $h$ is above the the graph of $i$ (the solution of $h(x) > i(x)$) as well as the $x$-coordinates of the intersection points of both graphs (where $h(x) = i(x)$). The combined sign chart is given on the left and the graphs are on the right.




□

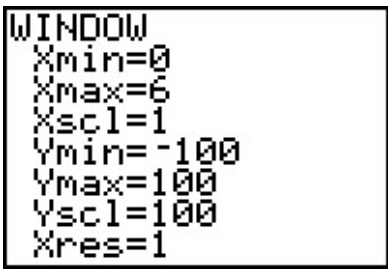
It is quite possible to encounter inequalities where the analytical methods developed so far will fail us. In this case, we resort to using the graphing calculator to approximate the solution, as the next example illustrates.

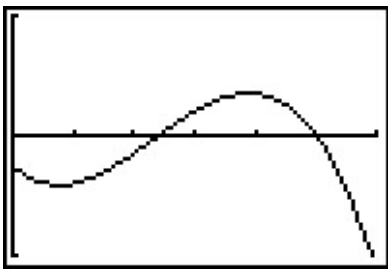
EXAMPLE 3.4.5. Suppose the revenue $R$, in thousands of dollars, from producing and selling $x$ hundred LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $x \geq 0$, while the cost, in thousands of dollars, to produce $x$ hundred LCD TVs is given by $C(x) = 200x + 25$ for $x \geq 0$. How many TVs, to the nearest TV, should be produced to make a profit?

SOLUTION. Recall that profit = revenue − cost. If we let $P$ denote the profit, in thousands of dollars, which results from producing and selling $x$ hundred TVs then

$$P(x) = R(x) - C(x) = \left(-5x^3 + 35x^2 + 155x\right) - (200x + 25) = -5x^3 + 35x^2 - 45x - 25,$$

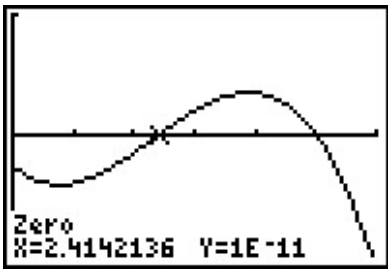
where $x \geq 0$. If we want to make a profit, then we need to solve $P(x) > 0$; in other words, $-5x^3 + 35x^2 - 45x - 25 > 0$. We have yet to discuss how to go about finding the zeros of $P$, let alone making a sign diagram for such an animal,[4] as such we resort to the graphing calculator. After finding a suitable window, we get

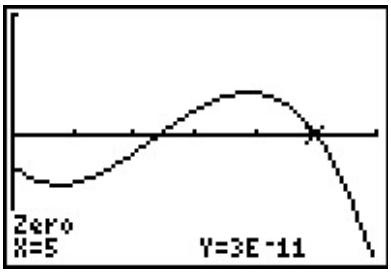


We are looking for the $x$ values for which $P(x) > 0$, that is, where the graph of $P$ is above the $x$-axis. We make use of the 'Zero' command and find two $x$-intercepts.

[4]The procedure, as we shall see in Chapter 4 is identical to what we have developed here.




We remember that $x$ denotes the number of TVs in **hundreds**, so if we are to find our solution using the calculator, we need our answer to two decimal places. The zero[5] $2.414\ldots$ corresponds to $241.4\ldots$ TVs. Since we can't make a fractional part of a TV, we round this up to 242 TVs.[7] The other zero seems dead on at 5, which corresponds to 500 TVs. Hence to make a profit, we should produce (and sell) between 242 and 499 TVs, inclusive. □

Our last example in the section demonstrates how inequalities can be used to describe regions in the plane, as we saw earlier in Section 1.2.

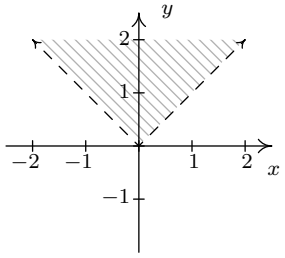
Example 3.4.6. Sketch the following relations.

1. $R = \{(x, y) : y > |x|\}$.
2. $S = \{(x, y) : y \leq 2 - x^2\}$.
3. $T = \{(x, y) : |x| < y \leq 2 - x^2\}$.

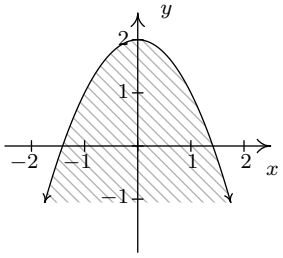
Solution.

1. The relation $R$ consists of all points $(x, y)$ whose $y$-coordinate is greater than $|x|$. If we graph $y = |x|$, then we want all of the points in the plane **above** the points on the graph. Dotting the graph of $y = |x|$ as we have done before to indicate the points on the graph itself are not in the relation, we get the shaded region below on the left.

2. For a point to be in $S$, its $y$-coordinate must be less than or equal to the $y$-coordinate on the parabola $y = 2 - x^2$. This is the set of all points **below** or **on** the parabola $y = 2 - x^2$.




The graph of $R$



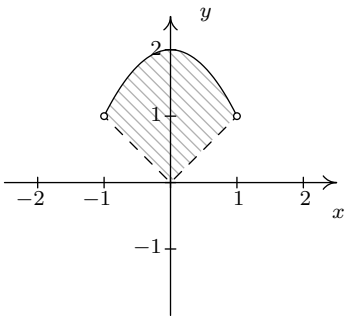

The graph of $S$

[5] Note the $y$-coordinates of the points here aren't registered as 0. They are expressed in Scientific Notation. For instance, $1E-11$ corresponds to 0.00000000001, which is pretty close in the calculator's eyes[6] to 0.

[6] but not a Mathematician's

[7] Notice that $P(241) < 0$ and $P(242) > 0$ so we need to round up to 242 in order to make a profit.

3. Finally, the relation $T$ takes the points whose $y$-coordinates satisfy both the conditions in $R$ and $S$. So we shade the region between $y = |x|$ and $y = 2 - x^2$, keeping those points on the parabola, but not the points on $y = |x|$. To get an accurate graph, we need to find where these two graphs intersect, so we set $|x| = 2 - x^2$. Proceeding as before, breaking this equation into cases, we get $x = -1, 1$. Graphing yields




The graph of $T$

□

### 3.4.1 Exercises

1. Solve the inequality. Express your answer in interval form.

   (a) $|3x - 5| \leq 4$
   (b) $|7x + 2| > 10$
   (c) $1 < |2x - 9| \leq 3$
   (d) $|-2x + 1| \geq x + 5$
   (e) $|x + 3| \geq |6x + 9|$
   (f) $x^2 + 2x - 3 \geq 0$
   (g) $16x^2 + 8x + 1 > 0$
   (h) $x^2 + 9 < 6x$
   (i) $9x^2 + 16 \geq 24x$
   (j) $x^2 + 4 \leq 4x$
   (k) $x^2 + 1 < 0$
   (l) $3x^2 \leq 11x + 4$
   (m) $x > x^2$
   (n) $2x^2 - 4x - 1 > 0$
   (o) $5x + 4 \leq 3x^2$
   (p) $2 \leq |x^2 - 9| < 9$
   (q) $x^2 \leq |4x - 3|$
   (r) $x^2 + x + 1 \geq 0$

2. Prove the second, third and fourth parts of Theorem 3.2.

3. If a slingshot is used to shoot a marble straight up into the air from 2 meters above the ground with an initial velocity of 30 meters per second, for what values of time $t$ will the marble be over 35 meters above the ground? (Refer to Exercise 5 in Section 3.3 for assistance if needed.) Round your answers to two decimal places.

4. What temperature values in degrees Celsius are equivalent to the temperature range $50^\circ F$ to $95^\circ F$? (Refer to Exercise 3 in Section 3.1 for assistance if needed.)

5. The surface area $S$ of a cube with edge length $x$ is given by $S(x) = 6x^2$ for $x > 0$. Suppose the cubes your company manufactures are supposed to have a surface area of exactly 42 square centimeters, but the machines you own are old and cannot always make a cube with the precise surface area desired. Write an inequality using absolute value that says the surface area of a given cube is no more than 3 square centimeters away (high or low) from the target of 42 square centimeters. Solve the inequality and express your answer in interval form.

6. Sketch the following relations.

   (a) $R = \{(x, y) : y \leq x - 1\}$
   (b) $R = \{(x, y) : y > x^2 + 1\}$
   (c) $R = \{(x, y) : -1 < y \leq 2x + 1\}$
   (d) $R = \{(x, y) : x^2 \leq y < x + 2\}$
   (e) $R = \{(x, y) : |x| - 4 < y < 2 - x\}$
   (f) $R = \{(x, y) : x^2 < y \leq |4x - 3|\}$

7. Suppose $f$ is a function, $L$ is a real number and $\varepsilon$ is a positive number. Discuss with your classmates what the inequality $|f(x) - L| < \varepsilon$ means algebraically and graphically.[8]

[8] Understanding this type of inequality is really important in Calculus.

### 3.4.2 ANSWERS

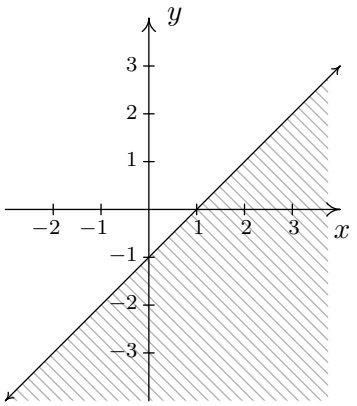
1. (a) $\left[\frac{1}{3}, 3\right]$
   (b) $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
   (c) $[3, 4) \cup (5, 6]$
   (d) $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
   (e) $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
   (f) $(-\infty, -3] \cup [1, \infty)$
   (g) $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
   (h) No solution
   (i) $(-\infty, \infty)$
   (j) $\{2\}$
   (k) No solution
   (l) $\left[-\frac{1}{3}, 4\right]$
   (m) $(0, 1)$
   (n) $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
   (o) $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$
   (p) $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup \left[-\sqrt{7}, 0\right) \cup \left(0, \sqrt{7}\right] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$
   (q) $\left[-2 - \sqrt{7}, -2 + \sqrt{7}\right] \cup [1, 3]$
   (r) $(-\infty, \infty)$

3. $1.44 < t < 4.68$

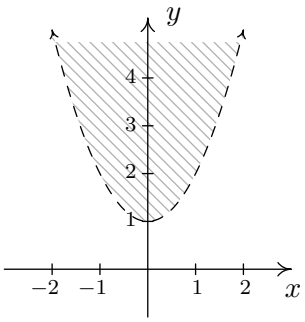
4. From our previous work $C(F) = \frac{5}{9}(F - 32)$ so $50 \leq F \leq 95$ becomes $10 \leq C \leq 35$.

5. The surface area could go as low as $39\text{cm}^2$ or as high as $45\text{cm}^2$ so we have $39 \leq S(x) \leq 45$. Using absolute value and the fact that $S(x) = 6x^2$ we get $|6x^2 - 42| \leq 3$. Solving this inequality yields $\left[\sqrt{\frac{13}{2}}, \sqrt{\frac{15}{2}}\right]$.

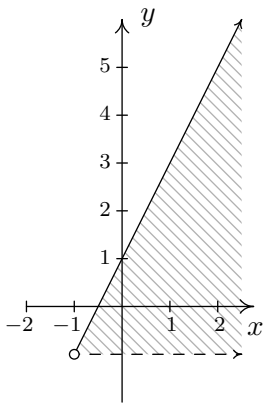
6. (a)



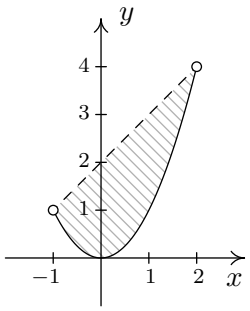

   (b)



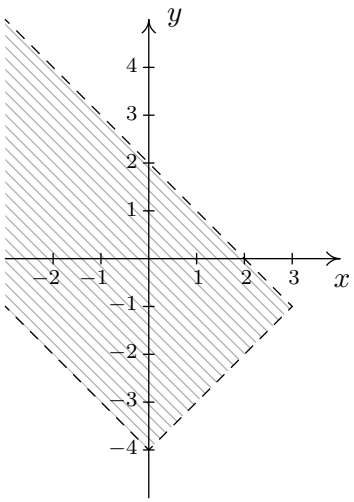
(c)



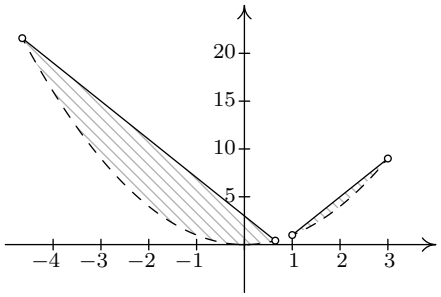

(d)




(e)




(f)



# Chapter 4

# Polynomial Functions

## 4.1 Graphs of Polynomials

Three of the families of functions studied thus far: constant, linear and quadratic, belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 4.1. A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where $a_0$, $a_1$. $\ldots a_n$ are real numbers and $n \geq 1$ is a natural number.[a] The domain of a polynomial function is $(-\infty, \infty)$.

[a] Recall this means $n$ is a 'counting number' $n = 1, 2, 3, \ldots$

There are several things about Definition 4.1 that may be off-putting or downright frightening. The best thing to do is look at an example. Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for $f$ as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with Definition 4.1, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$, and $a_0 = -5$. In other words, $a_5$ is the coefficient of $x^5$, $a_4$ is the coefficient of $x^4$, and so forth; the subscript on the $a$'s merely indicates to which power of $x$ the coefficient belongs. The business of restricting $n$ to be a natural number lets us focus on well-behaved algebraic animals.[1]

Example 4.1.1. Determine if the following functions are polynomials. Explain your reasoning.

[1] Enjoy this while it lasts. Before we're through with the book, you'll have been exposed to the most terrible of algebraic beasts. We will tame them all, in time.

1. $g(x) = \dfrac{4+x^3}{x}$
2. $p(x) = \dfrac{4x+x^3}{x}$
3. $q(x) = \dfrac{4x+x^3}{x^2+4}$
4. $f(x) = \sqrt[3]{x}$
5. $h(x) = |x|$
6. $z(x) = 0$

SOLUTION.

1. We note directly that the domain of $g(x) = \frac{x^3+4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, $g$ can't be a polynomial.

2. Even though $p(x) = \frac{x^3+4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in Definition 4.1, the domain of $p$, which, as you may recall, we determine **before** we simplify, excludes 0. Alas, $p$ is not a polynomial function for the same reason $g$ isn't.

3. After what happened with $p$ in the previous part, you may be a little shy about simplifying $q(x) = \frac{x^3+4x}{x^2+4}$ to $q(x) = x$, which certainly fits Definition 4.1. If we look at the domain of $q$ before we simplified, we see that it is, indeed, all real numbers. A function which can be written in the form of Definition 4.1 whose domain is all real numbers is, in fact, a polynomial.

4. We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\frac{1}{3}$ is not a natural number, $f$ is not a polynomial.

5. The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of $x$ (even though it can be written as a piecewise function involving polynomials.) As we shall see in this section, graphs of polynomials possess a quality[2] that the graph of $h$ does not.

6. There's nothing in Definition 4.1 which prevents all the coefficients $a_n$, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

[2] One which really relies on Calculus to verify.

Definition 4.2. Suppose $f$ is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$, we say
  - The natural number $n$ is called the **degree** of the polynomial $f$.
  - The term $a_n x^n$ is called the **leading term** of the polynomial $f$.
  - The real number $a_n$ is called the **leading coefficient** of the polynomial $f$.
  - The real number $a_0$ is called the **constant term** of the polynomial $f$.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say $f$ has degree 0.
- If $f(x) = 0$, we say $f$ has no degree.[a]

[a] Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials in Definition 4.2. Why not just lump them all together and, instead of forcing $n$ to be a natural number, $n = 1, 2, \ldots$, let $n$ be a whole number, $n = 0, 1, 2, \ldots$. We could unify all the cases, since, after all, isn't $a_0 x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$ and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not.[3] Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from Definition 4.2 is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

Example 4.1.2. Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x + x^3$
3. $h(x) = \dfrac{4 - x}{5}$
4. $p(x) = (2x - 1)^3(x - 2)(3x + 2)$

Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 4.2, and we see the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is $-5$.

[3] Technically, $0^0$ is an indeterminant form, which is a special case of being undefined. The authors realize this is beyond pedantry, but we wouldn't mention it if we didn't feel it was neccessary.

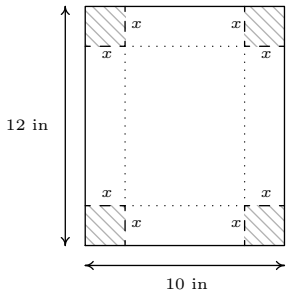
2. The form given in Definition 4.2 has the highest power of $x$ first. To that end, we re-write $g(x) = 12x + x^3 = x^3 + 12x$, and see the degree of $g$ is 3, the leading term is $x^3$, the leading coefficient is 1 and the constant term is 0.

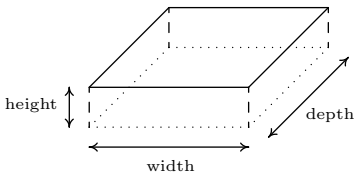
3. We need to rewrite the formula for $h$ so that it resembles the form given in Definition 4.2: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. We see the degree of $h$ is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

4. It may seem that we have some work ahead of us to get $p$ in the form of Definition 4.2. However, it is possible to glean the information requested about $p$ without multiplying out the entire expression $(2x-1)^3(x-2)(3x+2)$. The leading term of $p$ will be the term which has the highest power of $x$. The way to get this term is to multiply the terms with the highest power of $x$ from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of $p$ is $(2x)^3(x)(3x) = 24x^5$. This means the degree of $p$ is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

Our next example shows how polynomials of higher degree arise 'naturally'[4] in even the most basic geometric applications.

EXAMPLE 4.1.3. A box with no top is to be fashioned from a 10 inch × 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let $x$ denote the length of the side of the square which is removed from each corner.




1. Find the volume $V$ of the box as a function of $x$. Include an appropriate applied domain.

2. Use a graphing calculator to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

SOLUTION.

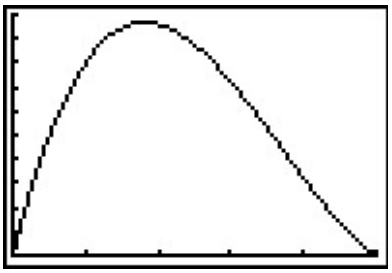
[4] this is a dangerous word...

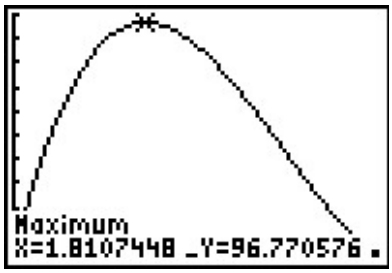
1. From Geometry, we know Volume = width × height × depth. The key is to now find each of these quantities in terms of $x$. From the figure, we see the height of the box is $x$ itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of $x$ inches from each corner leaves $10 - 2x$ inches for the width.[5] As for the depth, the cardboard is initially 12 inches long, so after cutting out $x$ inches from each side, we would have $12 - 2x$ inches remaining. As a function[6] of $x$, the volume is

$$V(x) = x(10 - 2x)(12 - 2x) = 4x^3 - 44x^2 + 120x.$$

   To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graphing calculator, we see the graph of $y = V(x)$ has a relative maximum. For $0 < x < 5$, this is also the absolute maximum. Using the 'Maximum' feature of the calculator, we get $x \approx 1.81$, $y \approx 96.77$. The height, $x \approx 1.81$ inches, the width, $10 - 2x \approx 6.38$ inches, and the depth $12 - 2x \approx 8.38$ inches. The $y$-coordinate is the maximum volume, which is approximately 96.77 cubic inches (also written $\text{in}^3$).





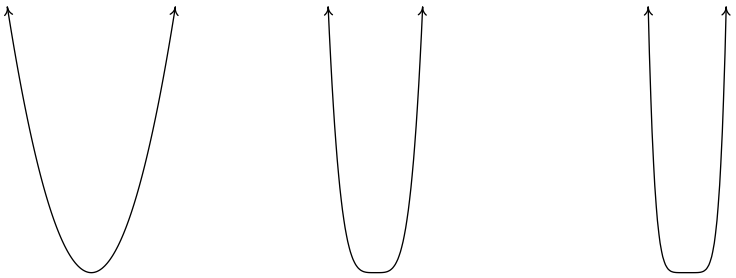

□

In order to solve Example 4.1.3, we made good use of the graph of the polynomial $y = V(x)$. So we ought to turn our attention to graphs of polynomials in general. Below are the graphs of $y = x^2$, $y = x^4$, and $y = x^6$, side-by-side. We have omitted the axes so we can see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand,[7] you will see why.

[5] There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out $x$ inches would leave $10 - 2x$ inches.

[6] When we write $V(x)$, it is in the context of function notation, not the volume $V$ times the quantity $x$.

[7] Make sure you choose some $x$-values between $-1$ and 1.




All of these functions are even, (Do you remember how to show this?) and it is exactly because the exponent is even.[8] One of the most important features of these functions which we can be seen graphically is their **end behavior**. The end behavior of a function is a way to describe what is happening to the function values as the $x$ values approach the 'ends' of the $x$-axis:[9] that is, as they become small without bound[10] (written $x \to -\infty$) and, on the flip side, as they become large without bound[11] (written $x \to \infty$). For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into $f$ to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behavior, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behavior of $f$ as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$. The same can be said for any function of the form $f(x) = x^n$ where $n$ is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the $x$-axis,[12] we have

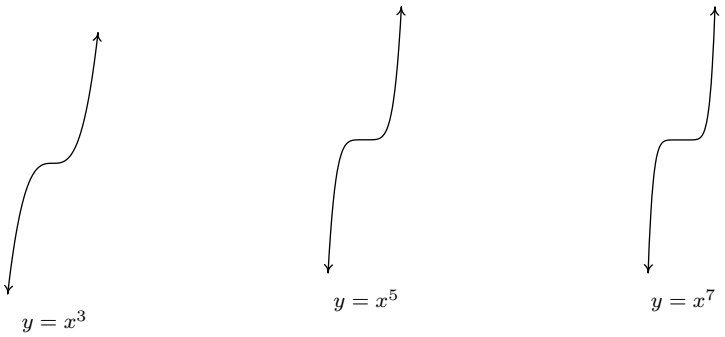
**End Behavior of functions $f(x) = ax^n$, $n$ even.**

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n$ is an even natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$ is an odd natural number.[13] Below we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd.[14] The end behavior of these functions is all the same, with

[8] Herein lies one of the possible origins of the term 'even' when applied to functions.
[9] Of course, there are no ends to the $x$-axis.
[10] We think of $x$ as becoming a very large **negative** number far to the left of zero.
[11] We think of $x$ as moving far to the right of zero and becoming a very large **positive** number.
[12] See Theorems 2.4 and 2.5 in Section 2.5.
[13] We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.
[14] And are, perhaps, the inspiration for the moniker 'odd function'.

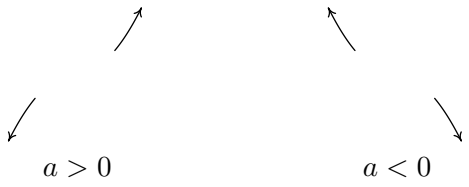
$f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.




As with the even degreed functions we studied earlier, we can generalize their end behavior.

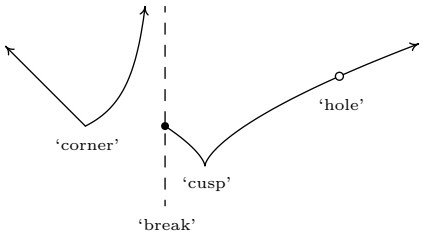
**End Behavior of functions $f(x) = ax^n$, $n$ odd.**

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

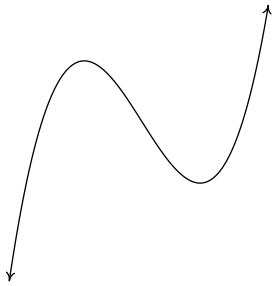



Despite having different end behavior, all functions of the form $f(x) = ax^n$ for natural numbers $n$ share two properties which help distinguish them from other animals in the algebra zoo: they are **continuous** and **smooth**. While these concepts are formally defined using Calculus,[15] informally, graphs of continuous functions have no 'breaks' or 'holes' in their graphs, and smooth functions have no 'sharp turns.' It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in graph on the right.

[15] In fact, if you take Calculus, you'll find that smooth functions are automatically continuous, so that saying 'polynomials are continuous and smooth' is redundant.



Pathologies not found on graphs of polynomials


The graph of a polynomial

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteric '$\vee$' shape, cannot be a polynomial. The notion of continuity is what allowed us to construct the sign diagram for quadratic inequalities as we did in Section 3.4. This last result is formalized in the following theorem.

THEOREM 4.1. **The Intermediate Value Theorem (Polynomial Zero Version):** If $f$ is a polynomial where $f(a)$ and $f(b)$ have different signs, then $f$ has at least one zero between $x = a$ and $x = b$; that is, for at least one real number $c$ such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most oft used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense, since it says, geometrically, that the graph of a polynomial function cannot be above the $x$-axis at one point and below the $x$-axis at another point without crossing the $x$-axis somewhere in between. The following example uses the Intermediate Value Theorem to establish a fact that that most students take for granted. Many students, and sadly some instructors, will find it silly.

EXAMPLE 4.1.4. Use the Intermediate Value Theorem to establish that $\sqrt{2}$ is a real number.

SOLUTION. Consider the polynomial function $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(3) = 7$. Since $f(1)$ and $f(3)$ have different signs, the Intermediate Value Theorem guarantees us a real number $c$ between 1 and 3 with $f(c) = 0$. If $c^2 - 2 = 0$ then $c = \pm\sqrt{2}$. Since $c$ is between 1 and 3, $c$ is positive, so $c = \sqrt{2}$. □

Our primary use of the Intermediate Value Theorem is in the construction of sign diagrams, as in Section 3.4, since it guarantees us that polynomial functions are always positive (+) or always negative (−) on intervals which do not contain any of its zeros. The general algorithm for polynomials is given below.

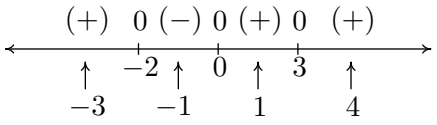
**Steps for Constructing a Sign Diagram for a Polynomial Function**

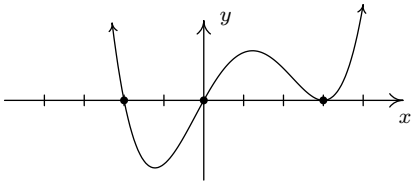
Suppose $f$ is a polynomial function.

1. Find the zeros of $f$ and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine the sign of $f(x)$ for each test value in step 2, and write that sign above the corresponding interval.

Example 4.1.5. Construct a sign diagram for $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$. Use it to give a rough sketch of the graph of $y = f(x)$.

Solution. First, we find the zeros of $f$ by solving $x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$. We get $x = 0$, $x = 3$, and $x = -2$. (The equation $x^2 + 1 = 0$ produces no real solutions.) These three points divide the real number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. We select the test values $x = -3$, $x = -1$, $x = 1$, and $x = 4$. We find $f(-3)$ is $(+)$, $f(-1)$ is $(-)$ and $f(1)$ is $(+)$ as is $f(4)$. Wherever $f$ is $(+)$, its graph is above the $x$-axis; wherever $f$ is $(-)$, its graph is below the $x$-axis. The $x$-intercepts of the graph of $f$ are $(-2, 0)$, $(0, 0)$ and $(3, 0)$. Knowing $f$ is smooth and continuous allows us to sketch its graph.




A sketch of $y = f(x)$

□

A couple of notes about the Example 4.1.5 are in order. First, note that we purposefully did not label the $y$-axis in the sketch of the graph of $y = f(x)$. This is because the sign diagram gives us the zeros and the relative position of the graph - it doesn't give us any information as to how high or low the graph strays from the $x$-axis. Furthermore, as we have mentioned earlier in the text, without Calculus, the values of the relative maximum and minimum can only be found approximately using a calculator. If we took the time to find the leading term of $f$, we would find it to be $x^8$. Looking at the end behavior of $f$, we notice it matches the end behavior of $y = x^8$. This is no accident, as we find out in the next theorem.

Theorem 4.2. **End Behavior for Polynomial Functions:** The end behavior of a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_n x^n$.

To see why Theorem 4.2 is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behavior, we look to see the behavior of $f$ as $x \to \pm\infty$. Since we're concerned with $x$'s far down the $x$-axis, we are far away from $x = 0$ and so can rewrite $f(x)$ for these values of $x$ as

$$f(x) = 4x^3\left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

As $x$ becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

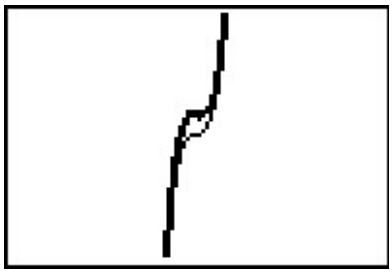
| $x$ | $\frac{1}{4x^2}$ | $\frac{5}{4x^3}$ |
|---|---|---|
| $-1000$ | 0.00000025 | $-0.00000000125$ |
| $-100$ | 0.000025 | $-0.00000125$ |
| $-10$ | 0.0025 | $-0.00125$ |
| 10 | 0.0025 | 0.00125 |
| 100 | 0.000025 | 0.00000125 |
| 1000 | 0.00000025 | 0.00000000125 |

In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3\,(1 - 0 + 0) = 4x^3$, which is the leading term of $f$. The formal proof of Theorem 4.2 works in much the same way. Factoring out the leading term leaves

$$f(x) = a_n x^n\left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

As $x \to \pm\infty$, any term with an $x$ in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_n x^n$. Geometrically, Theorem 4.2 says that if we graph $y = f(x)$, say, using a graphing calculator, and continue to 'zoom out,' the graph of it and its leading term become indistinguishable. Below are the graphs of $y = 4x^3 - x + 5$ (the thicker line) and $y = 4x^3$ (the thinner line) in two different windows.

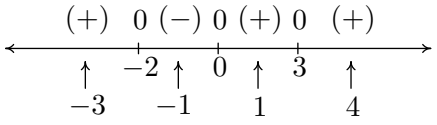
A view 'close' to the origin.

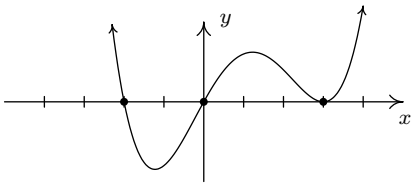


A 'zoomed out' view.

Let's return to the function in Example 4.1.5, $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, whose sign diagram and graph are reproduced below for reference. Theorem 4.2 tells us that the end behavior

is the same as that of its leading term, $x^8$. This tells us that the graph of $y = f(x)$ starts and ends above the $x$-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate $f$ at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate $f$ at the other test values? What we would really need to know is how the function behaves near its zeros – does it cross through the $x$-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$. From the sign diagram, the graph of $f$ will cross the $x$-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.




A sketch of $y = f(x)$

Fortunately, $f$ was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, $x^3$, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is **squared** kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and $x^3$, we note they are both odd - so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors change which results in the sign of the function value changing. This is the key to the behavior of the function near the zeros. We need a definition and then a theorem.

Definition 4.3. Suppose $f$ is a polynomial function and $m$ is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x = c$ is a zero of **multiplicity** $m$.

Hence, rewriting $f(x) = x^3(x-3)^2(x+2)$ as $f(x) = (x-0)^3(x-3)^2(x-(-2))^1$, we see that $x = 0$ is a zero of multiplicity 3, $x = 3$ is a zero of multiplicity 2, and $x = -2$ is a zero of multiplicity 1.

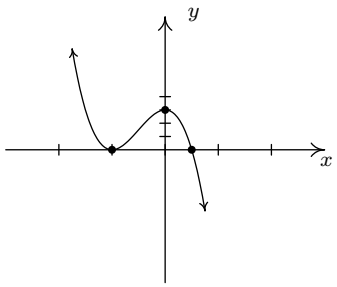
THEOREM 4.3. **The Role of Multiplicity:** Suppose $f$ is a polynomial function and $x = c$ is a zero of multiplicity $m$.

- If $m$ is even, the graph of $y = f(x)$ touches and rebounds from the $x$-axis as $(c, 0)$.
- If $m$ is odd, the graph of $y = f(x)$ crosses through the $x$-axis as $(c, 0)$.

Our last example shows how end behavior and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 4.1.6. Sketch the graph of $f(x) = -3(2x-1)(x+1)^2$ using end behavior and the multiplicity of its zeros.

Solution. The end behavior of the graph of $f$ will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2 = -6x^3$. This tells us the graph will start above the $x$-axis, in Quadrant II, and finish below the $x$-axis, in Quadrant IV. Next, we find the zeros of $f$. Fortunately for us, $f$ is factored.[16] Setting each factor equal to zero gives is $x = \frac{1}{2}$ and $x = -1$ as zeros. To find the multiplicity of $x = \frac{1}{2}$ we note that it corresponds to the factor $(2x-1)$. This isn't strictly in the form required in Definition 4.3. If we factor out the 2, however, we get $(2x-1) = 2\left(x - \frac{1}{2}\right)$, and we see the multiplicity of $x = \frac{1}{2}$ is 1. Since 1 is an odd number, we know from Theorem 4.3 that the graph of $f$ will cross through the $x$-axis at $\left(\frac{1}{2}, 0\right)$. Since the zero $x = -1$ corresponds to the factor $(x+1)^2 = (x-(-1))^2$, we see its multiplicity is 2 which is an even number. As such, the graph of $f$ will touch and rebound from the $x$-axis at $(-1, 0)$. Though we're not asked to, we can find the $y$-intercept by finding $f(0) = -3(2(0)-1)(0+1)^2 = 3$. Thus $(0, 3)$ is an additional point on the graph. Putting this together gives us the graph below.




□

[16] Obtaining the factored form of a polynomial is the main focus of the next few sections.

### 4.1.1 EXERCISES

1. For each polynomial given below, find the degree, the leading term, the leading coefficient, the constant term and the end behavior.

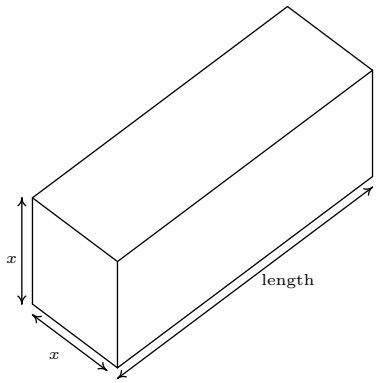
   (a) $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
   (b) $p(t) = -t^2(3 - 5t)(t^2 + t + 4)$
   (c) $Z(b) = 42b - b^3$
   (d) $s(t) = -4.9t^2 + v_0 t + s_0$
   (e) $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)$
   (f) $q(r) = 1 - 16r^4$

2. For each polynomial given below, find its real zeros and their corresponding multiplicities. Use this information along with a sign chart to provide a rough sketch of the graph of the polynomial.

   (a) $a(x) = x(x + 2)^2$
   (b) $F(x) = x^3(x + 2)^2$
   (c) $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)$
   (d) $Z(b) = b(42 - b^2)$
   (e) $Q(x) = (x + 5)^2(x - 3)^4$
   (f) $g(x) = x(x + 2)^3$

3. According to US Postal regulations, a rectangular shipping box must satisfy the inequality "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.[17] Let's assume we have a closed rectangular box with a square face of side length $x$ as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

   (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth = 130 inches, express the length of the box in terms of $x$ and then express the volume, $V$, of the box in terms of $x$.
   (b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.
   (c) Repeat parts 3a and 3b if the box is shipped using "other services".




[17] See here for details.

4. Use transformations to sketch the graphs of the following polynomials.

   (a) $f(x) = (x+2)^3 + 1$
   (b) $g(x) = (x+2)^4 + 1$
   (c) $h(x) = -x^5 - 3$
   (d) $j(x) = 2 - 3(x-1)^4$

5. Use the Intermediate Value Theorem to find intervals of length 1 which contain the real zeros of $f(x) = x^3 - 9x + 5$.

6. The original function used to model the cost of producing $x$ PortaBoys Game Systems given in Example 3.1.5 was $C(x) = 80x + 150$. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy revised their cost function using a cubic polynomial. The new cost of producing $x$ PortaBoys is given by $C(x) = .03x^3 - 4.5x^2 + 225x + 250$. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Find the production level $x$ that maximizes the profit made by producing and selling $x$ PortaBoys.

7. Here is a chart of the number of hours of daylight they get on the 21st of each month in Fairbanks, Alaska based on the 2009 sunrise and sunset data found on the U.S. Naval Observatory website. We let $x = 1$ represent January 21, 2009, $x = 2$ represent February 21, 2009, and so on.

| Month Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Hours of Daylight | 5.8 | 9.3 | 12.4 | 15.9 | 19.4 | 21.8 | 19.4 | 15.6 | 12.4 | 9.1 | 5.6 | 3.3 |

   Find cubic (third degree) and quartic (fourth degree) polynomials which model this data and comment on the goodness of fit for each. What can we say about using either model to make predictions about the year 2020? (Hint: Think about the end behavior of polynomials.) Use the models to see how many hours of daylight they got on your birthday and then check the website to see how accurate the models are. Knowing that Sasquatch are largely nocturnal, what days of the year according to your models are going to allow for at least 14 hours of darkness for field research on the elusive creatures?

8. An electric circuit is built with a variable resistor installed. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, $mW$) is given in the table below. [18]

| Resistance: $(k\Omega)$ | 1.012 | 2.199 | 3.275 | 4.676 | 6.805 | 9.975 |
|---|---|---|---|---|---|---|
| Power: $(mW)$ | 1.063 | 1.496 | 1.610 | 1.613 | 1.505 | 1.314 |

[18] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

(a) Make a scatter diagram of the data using the Resistance as the independent variable and Power as the dependent variable.

(b) Use your calculator to find quadratic (2nd degree), cubic (3rd degree) and quartic (4th degree) regression models for the data and judge the reasonableness of each.

(c) For each of the models found above, find the predicted maximum power that can be delivered to the load. What is the corresponding resistance value?

(d) Discuss with your classmates the limitations of these models - in particular, discuss the end behavior of each.

9. Show that the end behavior of a linear function $f(x) = mx + b$ is as it should be according to the results we've established in the section for polynomials of odd degree. (That is, show that the graph of a linear function is "up on one side and down on the other" just like the graph of $y = a_n x^n$ for odd numbers $n$.)

10. There is one subtlety about the role of multiplicity that we need to discuss further; specifically we need to see 'how' the graph crosses the $x$-axis at a zero of odd multiplicity. In the section, we deliberately excluded the function $f(x) = x$ from the discussion of the end behavior of $f(x) = x^n$ for odd numbers $n$ and we said at the time that it was due to the fact that $f(x) = x$ didn't fit the pattern we were trying to establish. You just showed in the previous exercise that the end behavior of a linear function behaves like every other polynomial of odd degree, so what doesn't $f(x) = x$ do that $g(x) = x^3$ does? It's the 'flattening' for values of $x$ near zero. It is this local behavior that will distinguish between a zero of multiplicity 1 and one of higher odd multiplicity. Look again closely at the graphs of $a(x) = x(x+2)^2$ and $F(x) = x^3(x+2)^2$ from Exercise 2. Discuss with your classmates how the graphs are fundamentally different at the origin. It might help to use a graphing calculator to zoom in on the origin to see the different crossing behavior. Also compare the behavior of $a(x) = x(x+2)^2$ to that of $g(x) = x(x+2)^3$ near the point $(-2, 0)$. What do you predict will happen at the zeros of $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$?

11. Here are a few other questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree $n$ have? How few local extrema can it have?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an $x$-intercept yield a local extrema? Can it yield an absolute extrema?

(h) If the $y$-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

### 4.1.2 Answers

1. (a) $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
   Degree 17
   Leading term $\sqrt{3}x^{17}$
   Leading coefficient $\sqrt{3}$
   Constant term $\frac{1}{3}$
   As $x \to -\infty,\ f(x) \to -\infty$
   As $x \to \infty,\ f(x) \to \infty$

   (b) $p(t) = -t^2(3-5t)(t^2+t+4)$
   Degree 5
   Leading term $5t^5$
   Leading coefficient 5
   Constant term 0
   As $t \to -\infty,\ p(t) \to -\infty$
   As $t \to \infty,\ p(t) \to \infty$

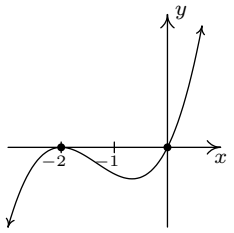
   (c) $Z(b) = 42b - b^3$
   Degree 3
   Leading term $-b^3$
   Leading coefficient $-1$
   Constant term 0
   As $b \to -\infty,\ Z(b) \to \infty$
   As $b \to \infty,\ Z(b) \to -\infty$

   (d) $s(t) = -4.9t^2 + v_0 t + s_0$
   Degree 2
   Leading term $-4.9t^2$
   Leading coefficient $-4.9$
   Constant term $s_0$
   As $t \to -\infty,\ s(t) \to -\infty$
   As $t \to \infty,\ s(t) \to -\infty$

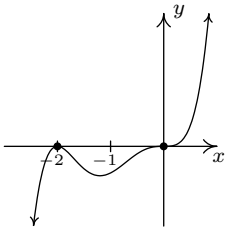
   (e) $P(x) = (x-1)(x-2)(x-3)(x-4)$
   Degree 4
   Leading term $x^4$
   Leading coefficient 1
   Constant term 24
   As $x \to -\infty,\ P(x) \to \infty$
   As $x \to \infty,\ P(x) \to \infty$

   (f) $q(r) = 1 - 16r^4$
   Degree 4
   Leading term $-16r^4$
   Leading coefficient $-16$
   Constant term 1
   As $r \to -\infty,\ q(r) \to -\infty$
   As $r \to \infty,\ q(r) \to -\infty$

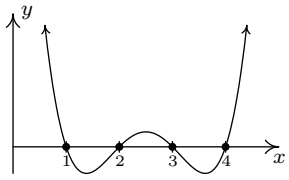
2. (a) $a(x) = x(x+2)^2$
   $x = 0$ multiplicity 1
   $x = -2$ multiplicity 2



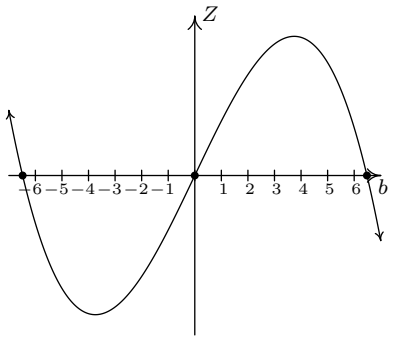

   (b) $F(x) = x^3(x+2)^2$
   $x = 0$ multiplicity 3
   $x = -2$ multiplicity 2



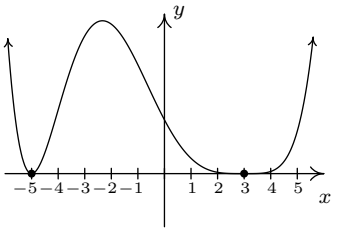
(c) $P(x) = (x-1)(x-2)(x-3)(x-4)$
$x = 1$ multiplicity 1
$x = 2$ multiplicity 1
$x = 3$ multiplicity 1
$x = 4$ multiplicity 1



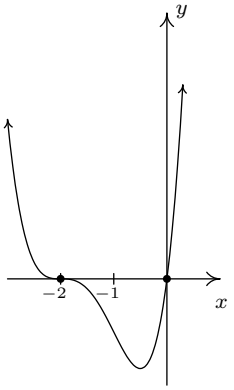

(d) $Z(b) = b(42 - b^2)$
$b = -\sqrt{42}$ multiplicity 1
$b = 0$ multiplicity 1
$b = \sqrt{42}$ multiplicity 1




(e) $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4




(f) $g(x) = x(x+2)^3$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 3

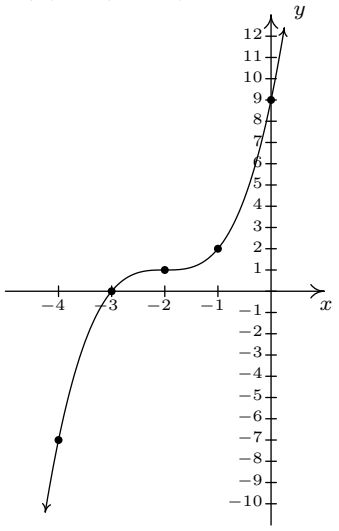



3. (a) Our ultimate goal is to maximize the volume, so we'll start with the maximum Length + Girth of 130. This means the length is $130 - 4x$. The volume of a rectangular box is always length × width × height so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

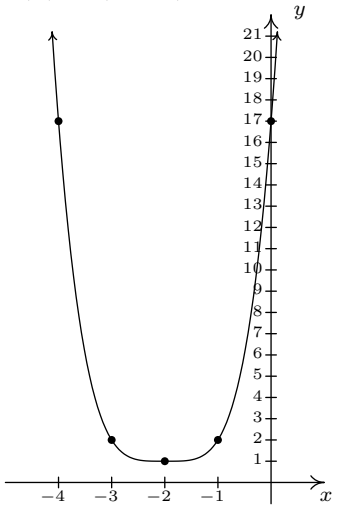
(b) Graphing $y = V(x)$ on $[0, 33] \times [0, 21000]$ shows a maximum at $(21.67, 20342.59)$ so the dimensions of the box with maximum volume are 21.67in. × 21.67in. × 43.32in. for a volume of $20342.59\text{in.}^3$.

(c) If we start with Length + Girth = 108 then the length is $108 - 4x$ and the volume is $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ on $[0, 27] \times [0, 11700]$ shows a maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. × 18.00in. × 36in. for a volume of $11664.00\text{in.}^3$. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all calculator results as approximations and list them as such.)

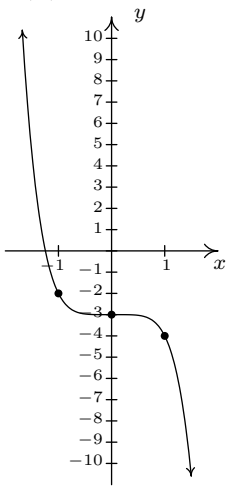
4. (a) $f(x) = (x+2)^3 + 1$



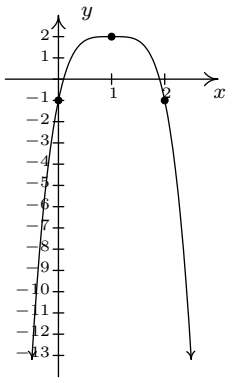

(b) $g(x) = (x+2)^4 + 1$




(c) $h(x) = -x^5 - 3$




(d) $j(x) = 2 - 3(x-1)^4$



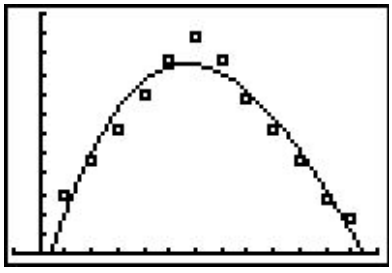

5. We have $f(-4) = -23$, $f(-3) = 5$, $f(0) = 5$, $f(1) = -3$, $f(2) = -5$ and $f(3) = 5$ so the Intermediate Value Theorem tells us that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3], [0, 1]$ and $[2, 3]$.

6. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

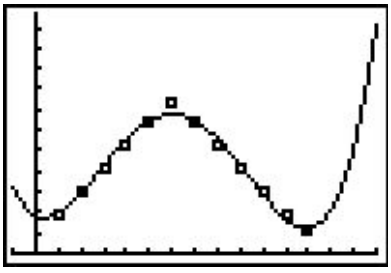
7. The cubic regression model is $p_3(x) = 0.0226x^3 - 0.9508x^2 + 8.615x - 3.446$. It has $R^2 = 0.93765$ which isn't bad. The graph of $y = p_3(x)$ in the viewing window $[-1, 13] \times [0, 24]$ along with the scatter plot is shown below on the left. Notice that $p_3$ hits the $x$-axis at about $x = 12.45$ making this a bad model for future predictions. To use the model to approximate the number of hours of sunlight on your birthday, you'll have to figure out what decimal value

of $x$ is close enough to your birthday and then plug it into the model. My (Jeff's) birthday is July 31 which is 10 days after July 21 ($x = 7$). Assuming 30 days in a month, I think $x = 7.33$ should work for my birthday and $p_3(7.33) \approx 17.5$. The website says there will be about 18.25 hours of daylight that day. To have 14 hours of darkness we need 10 hours of daylight. We see that $p_3(1.96) \approx 10$ and $p_3(10.05) \approx 10$ so it seems reasonable to say that we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to February 21, 2009 ($x = 2$) and then again from October 21,2009 ($x = 10$) to December 21, 2009 ($x = 12$).

The quartic regression model is $p_4(x) = 0.0144x^4 - 0.3507x^3 + 2.259x^2 - 1.571x + 5.513$. It has $R^2 = 0.98594$ which is good. The graph of $y = p_4(x)$ in the viewing window $[-1, 15] \times [0, 35]$ along with the scatter plot is shown below on the right. Notice that $p_4(15)$ is above 24 making this a bad model as well for future predictions. However, $p_4(7.33) \approx 18.71$ making it much better at predicting the hours of daylight on July 31 (my birthday). This model says we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to about March 1, 2009 ($x = 2.30$) and then again from October 10, 2009 ($x = 9.667$) to December 21, 2009 ($x = 12$).



$y = p_3(x)$

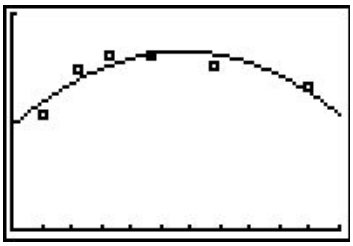


$y = p_4(x)$

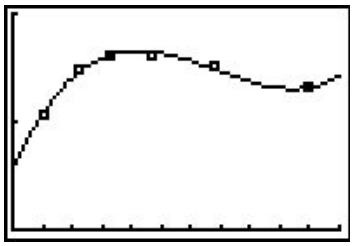
8. (a) The scatter plot is shown below with each of the three regression models.

   (b) The quadratic model is $P_2(x) = -0.02x^2 + 0.241x + 0.956$ with $R^2 = 0.77708$.
   The cubic model is $P_3(x) = 0.005x^3 - 0.103x^2 + 0.602x + 0.573$ with $R^2 = 0.98153$.
   The quartic model is $P_4(x) = -0.000969x^4 + 0.0253x^3 - 0.240x^2 + 0.944x + 0.330$ with $R^2 = 0.99929$.

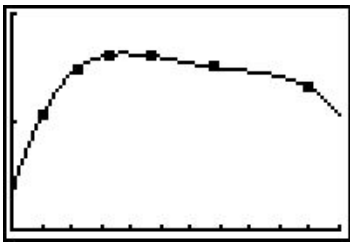
   (c) The maximums predicted by the three models are $P_2(5.737) \approx 1.648$, $P_3(4.232) \approx 1.657$ and $P_4(3.784) \approx 1.630$, respectively.



$y = P_2(x)$



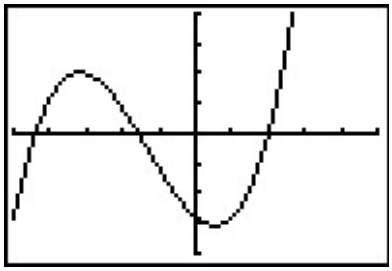
$y = P_3(x)$



$y = P_4(x)$

## 4.2 The Factor Theorem and The Remainder Theorem

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned[1] in Intermediate Algebra, this equation foils[2] us at every turn. If we graph $f$ using the graphing calculator, we get



The graph suggests that $x = 2$ is a zero, and we can verify $f(2) = 0$. The other two zeros seem to be less friendly, and, even though we could use the 'Zero' command to find decimal approximations for these, we seek a method to find the remaining zeros exactly. Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, it seems reasonable to expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{r|l}
 & x^2 + 6x + 7 \\
\hline
x-2 & x^3 + 4x^2 - 5x - 14 \\
 & \underline{-\left(x^3 - 2x^2\right)} \\
 & \quad 6x^2 - 5x \\
 & \quad \underline{-\left(6x^2 - 12x\right)} \\
 & \qquad 7x - 14 \\
 & \qquad \underline{-\,(7x - 14)} \\
 & \qquad\quad 0
\end{array}
$$

As you may recall, this means $x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$, and so to find the zeros of $f$, we now can solve $(x - 2)\left(x^2 + 6x + 7\right) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to generalize the technique applied here.

[1] and probably forgot
[2] pun intended

First up is a friendly reminder of what we can expect when we divide polynomials.

THEOREM 4.4. **Polynomial Division:** Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of $p$ is greater than or equal to the degree of $d$. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of $r$ is strictly less than the degree of $d$.

As you may recall, all of the polynomials in Theorem 4.4 have special names. The polynomial $p$ is called the **dividend**; $d$ is the **divisor**; $q$ is the **quotient**; $r$ is the **remainder**. If $r(x) = 0$ then $d$ is called a **factor** of $p$. The proof of Theorem 4.4 is usually relegated to a course in Abstract Algebra,[3] but we will use the result to establish two important facts which are the basis of the rest of the chapter.

THEOREM 4.5. **The Remainder Theorem:** Suppose $p$ is a polynomial of degree at least 1 and $c$ is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

The proof of Theorem 4.5 is a direct consequence of Theorem 4.4. When a polynomial is divided by $x - c$, the remainder is either 0 or has degree less than the degree of $x - c$. Since $x - c$ is degree 1, this means the degree of the remainder must be 0, which means the remainder is a constant. Hence, in either case, $p(x) = (x - c)\,q(x) + r$, where $r$, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, and so we get $r = p(c)$, as required. There is one last 'low hanging fruit'[4] to collect - it is an immediate consequence of The Remainder Theorem.

THEOREM 4.6. **The Factor Theorem:** Suppose $p$ is a nonzero polynomial. The real number $c$ is a zero of $p$ if and only if $(x - c)$ is a factor of $p(x)$.

The proof of The Factor Theorem is a consequence of what we already know. If $(x - c)$ is a factor of $p(x)$, this means $p(x) = (x - c)\,q(x)$ for some polynomial $q$. Hence, $p(c) = (c - c)\,q(c) = 0$, and so $c$ is a zero of $p$. Conversely, if $c$ is a zero of $p$, then $p(c) = 0$. In this case, The Remainder Theorem tells us the remainder when $p(x)$ is divided by $(x - c)$, namely $p(c)$, is 0, which means $(x - c)$ is a factor of $p$. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomials by quantities of the form $x - c$. Fortunately, people like

[3] Yes, Virginia, there are algebra courses more abstract than this one.

[4] Jeff hates this expression and Carl included it just to annoy him.

Ruffini and Horner have already blazed this trail. Let's take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the $-1$s.

$$\begin{array}{r|l} & x^2 + 6x + 7 \\ \hline x-2 & x^3 + 4x^2 - 5x - 14 \\ & \underline{-x^3 + 2x^2} \\ & \quad 6x^2 - 5x \\ & \quad \underline{-6x^2 + 12x} \\ & \qquad 7x - 14 \\ & \qquad \underline{-7x + 14} \\ & \qquad\quad 0 \end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and $-14$) aren't really necessary to recopy, and so we omit them, too.

$$\begin{array}{r|l} & x^2 + 6x + 7 \\ \hline x-2 & x^3 + 4x^2 - 5x - 14 \\ & \underline{\quad 2x^2 \quad\quad} \\ & \quad 6x^2 \\ & \quad\quad \underline{12x} \\ & \qquad 7x \\ & \qquad\quad \underline{14} \\ & \qquad\quad 0 \end{array}$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the

$x^3$ into the last row.

$$\begin{array}{rr|l} & & x^2+6x+7 \\ \hline x-2 & x^3+4x^2-5x-14 & \\ & 2x^2 \quad 12x \quad 14 & \\ \hline & x^3 \quad 6x^2 \quad 7x \quad 0 & \end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by $x$ and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the $x$ in the divisor, to determine our answer.

$$\begin{array}{rr} -2 & x^3+4x^2-5x-14 \\ & 2x^2 \quad 12x \quad 14 \\ \hline & x^3 \quad 6x^2 \quad 7x \quad 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$, and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, $x^2$, $6x$ and 7, respectively, by the $-2$ in $x-2$, then by $-1$ when we changed the subtraction to addition. Multiplying by $-2$ then by $-1$ is the same as multiplying by 2, and so we replace the $-2$ in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{rr} 2 & 1 \quad 4 \quad -5 \quad -14 \\ & \quad 2 \quad 12 \quad 14 \\ \hline & 1 \quad 6 \quad 7 \quad 0 \end{array}$$

We have constructed is the **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the $-5$ to get 7.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the $-14$ to get 0.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works **only** for these kinds of divisors.[5] Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Example 4.2.1. Use synthetic division to perform the following polynomial divisions. Find the quotient and the remainder polynomials, then write the dividend, quotient and remainder in the form given in Theorem 4.4.

[5] You'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(x^3 + 8\right) \div (x + 2)$
3. $\dfrac{4 - 8x - 12x^2}{2x - 3}$

SOLUTION.

1. When setting up the synthetic division tableau, we need to enter 0 for the coefficient of $x$ in the dividend. Doing so gives

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

   Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. According to Theorem 4.4, we have $5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$.

2. For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

   We get the quotient $q(x) = x^2 - 2x + 4$ and the remainder $r(x) = 0$. Relating the dividend, quotient and remainder gives $x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$.

3. To divide $4 - 8x - 12x^2$ by $2x - 3$, two things must be done. First, we write the dividend in descending powers of $x$ as $-12x^2 - 8x + 4$. Second, since synthetic division works only for factors of the form $x - c$, we factor $2x - 3$ as $2\left(x - \frac{3}{2}\right)$. Our strategy is to first divide $-12x^2 - 8x + 4$ by 2, to get $-6x^2 - 4x + 2$. Next, we divide by $\left(x - \frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

From this, we get $-6x^2 - 4x + 2 = \left(x - \frac{3}{2}\right)(-6x - 13) - \frac{35}{2}$. Multiplying both sides by 2 and distributing gives $-12x^2 - 8x + 4 = (2x - 3)(-6x - 13) - 35$. At this stage, we have written $-12x^2 - 8x + 4$ in the **form** $(2x - 3)q(x) + r(x)$, but how can we be sure the quotient polynomial is

$-6x-13$ and the remainder is $-35$? The answer is the word 'unique' in Theorem 4.4. The theorem states that there is only one way to decompose $-12x^2-8x+4$ into a multiple of $(2x-3)$ plus a constant term. Since we have found such a way, we can be sure it is the only way. □

The next example pulls together all of the concepts discussed in this section.

Example 4.2.2. Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.
2. Use the fact that $x = 1$ is a zero of $p$ to factor $p(x)$ and find all of the real zeros of $p$.

Solution.

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x-(-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of $x^2$ as 0, and proceed as above.

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

   According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. The Factor Theorem tells us that since $x = 1$ is a zero of $p$, $x-1$ is a factor of $p(x)$. To factor $p(x)$, we divide

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

   We get a remainder of 0 which verifies that, indeed, $p(1) = 0$. Our quotient polynomial is a second degree polynomial with coefficients 2, 2, and $-3$. So $q(x) = 2x^2 + 2x - 3$. Theorem 4.4 tells us $p(x) = (x-1)\left(2x^2+2x-3\right)$. To find the remaining real zeros of $p$, we need to solve $2x^2 + 2x - 3 = 0$ for $x$. Since this doesn't factor nicely, we use the quadratic formula to find that the remaining zeros are $x = \frac{-1 \pm \sqrt{7}}{2}$. □

In Section 4.1, we discussed the notion of the multiplicity of a zero. Roughly speaking, a zero with multiplicity 2 can be divided twice into a polynomial; multiplicity 3, three times and so on. This is illustrated in the next example.

EXAMPLE 4.2.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Given that $x = \frac{1}{2}$ is a zero of multiplicity 2, find all of the real zeros of $p$.

SOLUTION. We set up for synthetic division. Since we are told the multiplicity of $\frac{1}{2}$ is two, we continue our tableau and divide $\frac{1}{2}$ into the quotient polynomial

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline \frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\ & \downarrow & 2 & 0 & -6 & \\ \hline & 4 & 0 & -12 & \boxed{0} & \end{array}$$

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division tells us $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$. Combining these results, we have $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 - 12\right)$. To find the remaining zeros of $p$, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$. □

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common site in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula. Secondly, we found $x = \pm\sqrt{3}$ are zeros of $p$. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of $p$. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline \frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\ & \downarrow & 2 & 0 & -6 & \\ \hline \sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\ & \downarrow & 4\sqrt{3} & 12 & & \\ \hline -\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\ & \downarrow & -4\sqrt{3} & & & \\ \hline & 4 & \boxed{0} & & & \end{array}$$

This gives us $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2\left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)(4)$, or, when written

with the constant in front

$$p(x) = 4\left(x - \frac{1}{2}\right)^2\left(x - \sqrt{3}\right)\left(x - \left(-\sqrt{3}\right)\right)$$

We have shown that $p$ is a product of its leading term times linear factors of the form $(x-c)$ where $c$ are zeros of $p$. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section 10.1, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

Theorem 4.7. Suppose $f$ is a polynomial of degree $n$, $n \geq 1$. Then $f$ has at most $n$ real zeros, counting multiplicities.

Theorem 4.7 is a consequence of the Factor Theorem and polynomial multiplication. Every zero $c$ of $f$ gives us a factor of the form $(x-c)$ for $f(x)$. Since $f$ has degree $n$, there can be at most $n$ of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

**Connections Between Zeros, Factors and Graphs of Polynomial Functions**

Suppose $p$ is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number $c$ is a zero of $p$
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x-c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an $x$-intercept of the graph of $y = p(x)$

### 4.2.1 EXERCISES

1. (An Intermediate Algebra review exercise) Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

   (a) $(5x^4 - 3x^3 + 2x^2 - 1) \div (x^2 + 4)$
   (b) $(-x^5 + 7x^3 - x) \div (x^3 - x^2 + 1)$
   (c) $(9x^3 + 5) \div (2x - 3)$
   (d) $(4x^2 - x - 23) \div (x^2 - 1)$

2. Use synthetic division and the Remainder Theorem to test whether or not the given number is a zero of the polynomial $p(x) = 15x^5 - 121x^4 + 17x^3 - 73x^2 + 2x + 48$.

   (a) $c = -1$
   (b) $c = 8$
   (c) $c = \frac{1}{2}$
   (d) $c = \frac{2}{3}$
   (e) $c = 0$
   (f) $c = -\frac{3}{5}$

3. For each polynomial given below, you are given one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

   (a) $x^3 - 6x^2 + 11x - 6, \;\; c = 1$
   (b) $x^3 - 24x^2 + 192x - 512, \;\; c = 8$
   (c) $4x^4 - 28x^3 + 61x^2 - 42x + 9, \;\; c = \frac{1}{2}$
   (d) $3x^3 + 4x^2 - x - 2, \;\; c = \frac{2}{3}$
   (e) $x^4 - x^2, \;\; c = 0$
   (f) $x^2 - 2x - 2, \;\; c = 1 - \sqrt{3}$
   (g) $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324, \;\; c = -\frac{3}{5}$

4. Create a polynomial $p$ with the following attributes.

   - As $x \to -\infty$, $p(x) \to \infty$.
   - The point $(-2, 0)$ yields a local maximum.
   - The degree of $p$ is 5.
   - The point $(3, 0)$ is one of the $x$-intercepts of the graph of $p$.

5. Find a quadratic polynomial with integer coefficients which has $x = \dfrac{3}{5} \pm \dfrac{\sqrt{29}}{5}$ as its real zeros.

### 4.2.2 Answers

1. (a) $5x^4 - 3x^3 + 2x^2 - 1 = (x^2 + 4)(5x^2 - 3x - 18) + (12x + 71)$
   (b) $-x^5 + 7x^3 - x = (x^3 - x^2 + 1)(-x^2 - x + 6) + (7x^2 - 6)$
   (c) $9x^3 + 5 = (2x - 3)(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}) + \frac{283}{8}$
   (d) $4x^2 - x - 23 = (x^2 - 1)(4) + (-x - 19)$

2. (a) $p(-1) - 180$
   (b) $p(8) = 0$
   (c) $p(\frac{1}{2}) = \frac{825}{32}$
   (d) $p(\frac{2}{3}) = 0$
   (e) $p(0) = 48$
   (f) $p(-\frac{3}{5}) = 0$

3. (a) $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$
   (b) $x^3 - 24x^2 + 192x - 512 = (x - 8)^3$
   (c) $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4(x - \frac{1}{2})^2(x - 3)^2$
   (d) $3x^3 + 4x^2 - x - 2 = 3(x - \frac{2}{3})(x + 1)^2$
   (e) $x^4 - x^2 = x^2(x - 1)(x + 1)$
   (f) $x^2 - 2x - 2 = (x - (1 - \sqrt{3}))(x - (1 + \sqrt{3}))$
   (g) $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324 = 125(x + \frac{3}{5})^3(x - 6)(x + 2)$

4. Something like $p(x) = -(x + 2)^2(x - 3)(x + 3)(x - 4)$ will work.

5. $q(x) = 5x^2 - 6x - 4$

# Chapter 5

# Rational Functions

## 5.1 Introduction to Rational Functions

If we add, subtract or multiply polynomial functions according to the function arithmetic rules defined in Section 2.3, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. In this chapter we study **rational functions** - functions which are ratios of polynomials.

Definition 5.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, $r$ is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p$ and $q$ are polynomial functions[a]

[a] According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

As we recall from Section 2.2, we have domain issues anytime the denominator of a fraction is zero. In the example below, we review this concept as well as some of the arithmetic of rational expressions.

Example 5.1.1. Find the domain of the following rational functions. Write them in the form $\frac{p(x)}{q(x)}$ for polynomial functions $p$ and $q$ and simplify.

1. $f(x) = \dfrac{2x-1}{x+1}$
2. $g(x) = 2 - \dfrac{3}{x+1}$
3. $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$
4. $r(x) = \dfrac{2x^2-1}{x^2-1} \div \dfrac{3x-2}{x^2-1}$

SOLUTION.

1. To find the domain of $f$, we proceed as we did in Section 2.2: we find the zeros of the denominator and exclude them from the domain. Setting $x+1=0$ results in $x=-1$. Hence, our domain is $(-\infty,-1)\cup(-1,\infty)$. The expression $f(x)$ is already in the form requested and when we check for common factors among the numerator and denominator we find none, so we are done.

2. Proceeding as before, we determine the domain of $g$ by solving $x+1=0$. As before, we find the domain of $g$ is $(-\infty,-1)\cup(-1,\infty)$. To write $g(x)$ in the form requested, we need to get a common denominator

$$\begin{aligned} g(x) &= 2-\frac{3}{x+1} \\ &= \frac{2}{1}-\frac{3}{x+1} \\ &= \frac{(2)(x+1)}{(1)(x+1)}-\frac{3}{x+1} \\ &= \frac{(2x+2)-3}{x+1} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

This formula is also completely simplified.

3. The denominators in the formula for $h(x)$ are both $x^2-1$ whose zeros are $x=\pm 1$. As a result, the domain of $h$ is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. We now proceed to simplify $h(x)$. Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned} h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} \\ &= \frac{\left(2x^2-1\right)-(3x-2)}{x^2-1} \\ &= \frac{2x^2-1-3x+2}{x^2-1} \\ &= \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} \\ &= \frac{(2x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

4. To find the domain of $r$, it may help to temporarily rewrite $r(x)$ as

$$r(x) = \frac{\dfrac{2x^2-1}{x^2-1}}{\dfrac{3x-2}{x^2-1}}$$

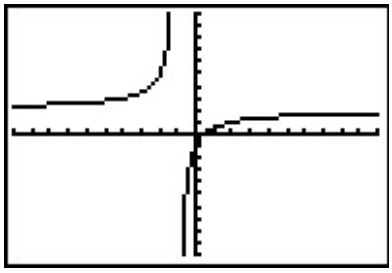
We need to set all of the denominators equal to zero which means we need to solve not only $x^2-1=0$, but also $\frac{3x-2}{x^2-1}=0$. We find $x=\pm 1$ for the former and $x=\frac{2}{3}$ for the latter. Our domain is $(-\infty,-1)\cup\left(-1,\frac{2}{3}\right)\cup\left(\frac{2}{3},1\right)\cup(1,\infty)$. We simplify $r(x)$ by rewriting the division as multiplication by the reciprocal and then simplifying

$$\begin{aligned} r(x) &= \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1} \\ &= \frac{2x^2-1}{x^2-1} \cdot \frac{x^2-1}{3x-2} \\ &= \frac{\left(2x^2-1\right)\left(x^2-1\right)}{\left(x^2-1\right)(3x-2)} \\ &= \frac{\left(2x^2-1\right)\cancel{\left(x^2-1\right)}}{\cancel{\left(x^2-1\right)}(3x-2)} \\ &= \frac{2x^2-1}{3x-2} \end{aligned}$$

□

A few remarks about Example 5.1.1 are in order. Note that the expressions for $f(x)$, $g(x)$ and $h(x)$ work out to be the same. However, only two of these functions are actually equal. Recall that functions are ultimately sets of ordered pairs,[1] and so for two functions to be equal, they need, among other things, to have the same domain. Since $f(x) = g(x)$ and $f$ and $g$ have the same domain, they are equal functions. Even though the formula $h(x)$ is the same as $f(x)$, the domain of $h$ is different than the domain of $f$, and thus they are different functions.

We now turn our attention to the graphs of rational functions. Consider the function $f(x) = \frac{2x-1}{x+1}$ from Example 5.1.1. Using a graphing calculator, we obtain



Two behaviors of the graph are worthy of further discussion. First, note that the graph appears to 'break' at $x = -1$. We know from our last example that $x = -1$ is not in the domain of $f$ which means $f(-1)$ is undefined. When we make a table of values to study the behavior of $f$ **near** $x = -1$ we see that we can get 'near' $x = -1$ from two directions. We can choose values a little less than $-1$, for example $x = -1.1$, $x = -1.01$, $x = -1.001$, and so on. These values are said to 'approach $-1$ from the **left**.' Similarly, the values $x = -0.9$, $x = -0.99$, $x = -0.999$, etc., are said to 'approach $-1$ from the **right**.' If we make two tables, we find that the numerical results confirm what we see graphically.

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-1.1$ | 32 | $(-1.1, 32)$ |
| $-1.01$ | 302 | $(-1.01, 302)$ |
| $-1.001$ | 3002 | $(-1.001, 3002)$ |
| $-1.0001$ | 30002 | $(-1.001, 30002)$ |

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-0.9$ | $-28$ | $(-0.9, -28)$ |
| $-0.99$ | $-298$ | $(-0.99, -298)$ |
| $-0.999$ | $-2998$ | $(-0.999, -2998)$ |
| $-0.9999$ | $-29998$ | $(-0.9999, -29998)$ |

As the $x$ values approach $-1$ from the left, the function values become larger and larger positive numbers.[2] We express this symbolically by stating as $x \to -1^-$, $f(x) \to \infty$. Similarly, using analogous notation, we conclude from the table that as $x \to -1^+$, $f(x) \to -\infty$. For this type of unbounded behavior, we say the graph of $y = f(x)$ has a **vertical asymptote** of $x = -1$. Roughly speaking, this means that near $x = -1$, the graph looks very much like the vertical line $x = -1$.

[1] You should review Sections 1.2 and 2.1 if this statement caught you off guard.
[2] We would need Calculus to confirm this analytically.

Another feature worthy of note about the graph of $y = f(x)$ is it seems to 'level off' on the left and right hand sides of the screen. This is a statement about the end behavior of the function. As we discussed in Section 4.1, the end behavior of a function is its behavior as $x$ as $x$ attains larger[3] and larger negative values without bound, $x \to -\infty$, and as $x$ becomes large without bound, $x \to \infty$. Making tables of values, we find

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-10$ | $\approx 2.3333$ | $\approx (-10, 2.3333)$ |
| $-100$ | $\approx 2.0303$ | $\approx (-100, 2.0303)$ |
| $-1000$ | $\approx 2.0030$ | $\approx (-1000, 2.0030)$ |
| $-10000$ | $\approx 2.0003$ | $\approx (-10000, 2.0003)$ |

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $10$ | $\approx 1.7273$ | $\approx (10, 1.7273)$ |
| $100$ | $\approx 1.9703$ | $\approx (100, 1.9703)$ |
| $1000$ | $\approx 1.9970$ | $\approx (1000, 1.9970)$ |
| $10000$ | $\approx 1.9997$ | $\approx (10000, 1.9997)$ |

From the tables, we see as $x \to -\infty$, $f(x) \to 2^+$ and as $x \to \infty$, $f(x) \to 2^-$. Here the '+' means 'from above' and the '$-$' means 'from below'. In this case, we say the graph of $y = f(x)$ has a **horizontal asymptote** of $y = 2$. This means that the end behavior of $f$ resembles the horizontal line $y = 2$, which explains the 'leveling off' behavior we see in the calculator's graph. We formalize the concepts of vertical and horizontal asymptotes in the following definitions.

Definition 5.2. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Definition 5.3. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, either $f(x) \to c^-$ or $f(x) \to c^+$.

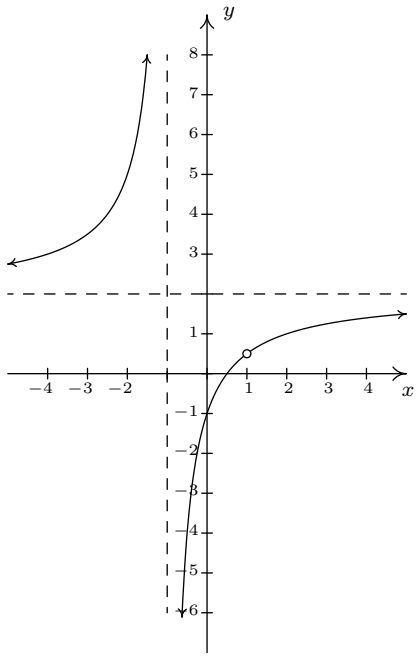
In our discussion following Example 5.1.1, we determined that, despite the fact that the formula for $h(x)$ reduced to the same formula as $f(x)$, the functions $f$ and $h$ are different, since $x = 1$ is in the domain of $f$, but $x = 1$ is not in the domain of $h$. If we graph $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$ using a graphing calculator, we are surprised to find that the graph looks identical to the graph of $y = f(x)$. There is a vertical asymptote at $x = -1$, but near $x = 1$, everything seem fine. Tables of values provide numerical evidence which supports the graphical observation.

[3]Here, the word 'larger' means larger in absolute value.

| $x$ | $h(x)$ | $(x, h(x))$ |
|---|---|---|
| 0.9 | $\approx 0.4210$ | $\approx (0.9, 0.4210)$ |
| 0.99 | $\approx 0.4925$ | $\approx (0.99, 0.4925)$ |
| 0.999 | $\approx 0.4992$ | $\approx (0.999, 0.4992)$ |
| 0.9999 | $\approx 0.4999$ | $\approx (0.9999, 0.4999)$ |

| $x$ | $h(x)$ | $(x, h(x))$ |
|---|---|---|
| 1.1 | $\approx 0.5714$ | $\approx (1.1, 0.5714)$ |
| 1.01 | $\approx 0.5075$ | $\approx (1.01, 0.5075)$ |
| 1.001 | $\approx 0.5007$ | $\approx (1.001, 0.5007)$ |
| 1.0001 | $\approx 0.5001$ | $\approx (1.0001, 0.5001)$ |

We see that as $x \to 1^-$, $h(x) \to 0.5^-$ and as $x \to 1^+$, $h(x) \to 0.5^+$. In other words, the points on the graph of $y = h(x)$ are approaching $(1, 0.5)$, but since $x = 1$ is not in the domain of $h$, it would be inaccurate to fill in a point at $(1, 0.5)$. As we've done in past sections when something like this occurs,[4] we put an open circle (also called a 'hole' in this case[5]) at $(1, 0.5)$. Below is a detailed graph of $y = h(x)$, with the vertical and horizontal asymptotes as dashed lines.




Neither $x = -1$ nor $x = 1$ are in the domain of $h$, yet we see the behavior of the graph of $y = h(x)$ is drastically different near these points. The reason for this lies in the second to last step when we simplified the formula for $h(x)$ in Example 5.1.1. We had $h(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$. The reason $x = -1$ is not in the domain of $h$ is because the factor $(x + 1)$ appears in the denominator of $h(x)$; similarly, $x = 1$ is not in the domain of $h$ because of the factor $(x - 1)$ in the denominator of $h(x)$. The major difference between these two factors is that $(x - 1)$ cancels with a factor in

[4] For instance, graphing piecewise defined functions in Section 2.4.

[5] Stay tuned. In Calculus, we will see how these 'holes' can be 'plugged' when embarking on a more advanced study of contintuity.

the numerator whereas $(x+1)$ does not. Loosely speaking, the trouble caused by $(x-1)$ in the denominator is canceled away while the factor $(x+1)$ remains to cause mischief. This is why the graph of $y = h(x)$ has a vertical asymptote at $x = -1$ but only a hole at $x = 1$. These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

Theorem 5.1. **Location of Vertical Asymptotes and Holes:**[a] Suppose $r$ is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where $p$ and $q$ have no common zeros.[b] Let $c$ be a real number which is not in the domain of $r$.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'
[b] In other words, $r(x)$ is in lowest terms.

In English, Theorem 5.1 says if $x = c$ is not in the domain of $r$ but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, we have a vertical asymptote.

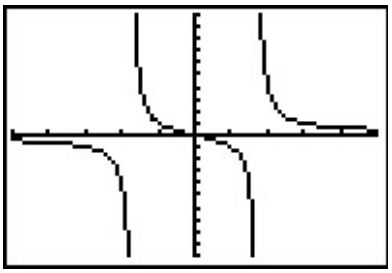
Example 5.1.2. Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions. Verify your answers using a graphing calculator.

1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$
3. $h(x) = \dfrac{x^2 - x - 6}{x^2 + 9}$
4. $r(x) = \dfrac{x^2 - x - 6}{x^2 + 4x + 4}$

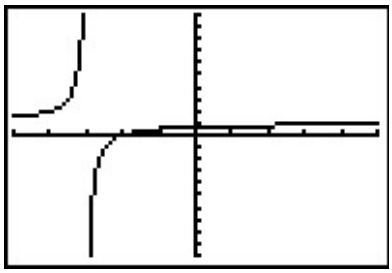
Solution.

1. To use Theorem 5.1, we first find all of the real numbers which aren't in the domain of $f$. To do so, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms, there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. The calculator verifies this claim.

2. Solving $x^2 - 9 = 0$ gives $x = \pm 3$. In lowest terms $g(x) = \frac{x^2-x-6}{x^2-9} = \frac{(x-3)(x+2)}{(x-3)(x+3)} = \frac{x+2}{x+3}$. Since $x = -3$ continues to make trouble in the denominator, we know the line $x = -3$ is a vertical asymptote of the graph of $y = g(x)$. Since $x = 3$ no longer produces a 0 in the denominator, we have a hole at $x = 3$. To find the $y$-coordinate of the hole, we substitute $x = 3$ into $\frac{x+2}{x+3}$ and find the hole is at $\left(3, \frac{5}{6}\right)$. When we graph $y = g(x)$ using a calculator, we clearly see the vertical asymptote at $x = -3$, but everything seems calm near $x = 3$.



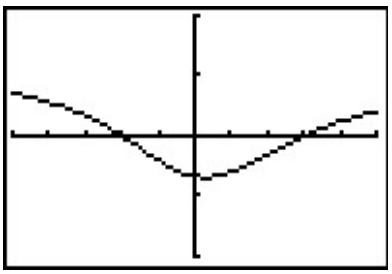
The graph of $y = f(x)$



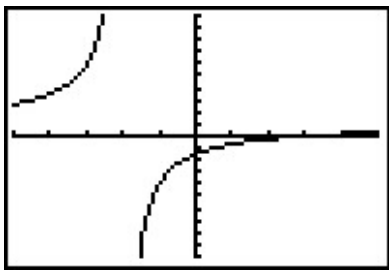
The graph of $y = g(x)$

3. The domain of $h$ is all real numbers, since $x^2 + 9 = 0$ has no real solutions. Accordingly, the graph of $y = h(x)$ is devoid of both vertical asymptotes and holes.

4. Setting $x^2 + 4x + 4 = 0$ gives us $x = -2$ as the only real number of concern. Simplifying, we see $r(x) = \frac{x^2-x-6}{x^2+4x+4} = \frac{(x-3)(x+2)}{(x+2)^2} = \frac{x-3}{x+2}$. Since $x = -2$ continues to produce a 0 in the denominator of the reduced function, we know $x = -2$ is a vertical asymptote to the graph, which the calculator confirms.



The graph of $y = h(x)$



The graph of $y = r(x)$

□

Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[6] The unfortunate name will make sense shortly.

EXAMPLE 5.1.3. A mathematical model for the population $P$, in thousands, of a certain species of bacteria, $t$ days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.

2. When will the population reach 100,000?

3. Determine the behavior of $P$ as $t \to 5^-$. Interpret this result graphically and within the context of the problem.

[6] This is a class of Calculus equations in which a population grows very rapidly.

Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$, which means 4000 bacteria are initially introduced into the environment.

2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in **thousands**. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Substituting for $P(t)$ gives the equation $\frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

3. To determine the behavior of $P$ as $t \to 5^-$, we can make a table

| $t$ | $P(t)$ |
|---|---|
| 4.9 | 10000 |
| 4.99 | 1000000 |
| 4.999 | 100000000 |
| 4.9999 | 10000000000 |

In other words, as $t \to 5^-$, $P(t) \to \infty$. Graphically, the line $t = 5$ is a vertical asymptote of the graph of $y = P(t)$. Physically, this means the population of bacteria is increasing without bound as we near 5 days, which cannot physically happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse. □

Now that we have thoroughly investigated vertical asymptotes, we now turn our attention to horizontal asymptotes. The next theorem tells us when to expect horizontal asymptotes.

THEOREM 5.2. **Location of Horizontal Asymptotes:** Suppose $r$ is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where $p$ and $q$ are polynomial functions with leading coefficients $a$ and $b$, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a] The use of the definite article will be justified momentarily.

Like Theorem 5.1, Theorem 5.2 is proved using Calculus. Nevertheless, we can understand the idea behind it using our example $f(x) = \frac{2x-1}{x+1}$. If we interpret $f(x)$ as a division problem, $(2x-1) \div (x+1)$, we find the quotient is 2 with a remainder of $-3$. Using what we know about polynomial division, specifically Theorem 4.4, we get $2x - 1 = 2(x+1) - 3$. Dividing both sides by $(x+1)$ gives $\frac{2x-1}{x+1} = 2 - \frac{3}{x+1}$. (You may remember this as the formula for $g(x)$ in Example 5.1.1.) As $x$ becomes unbounded in either direction, the quantity $\frac{3}{x+1}$ gets closer and closer to zero so that the values of $f(x)$ become closer and closer to 2. In symbols, as $x \to \pm\infty$, $f(x) \to 2$, and we have the result.[7] Notice that the graph gets close to the same $y$ value as $x \to -\infty$ or $x \to \infty$. This means that the graph can have only one horizontal asymptote if it is going to have one at all. Thus we were justified in using 'the' in the previous theorem. (By the way, using long division to determine the asymptote will serve us well in the next section so you might want to review that topic.)

Alternatively, we can use what we know about end behavior of polynomials to help us understand this theorem. From Theorem 4.2, we know the end behavior of a polynomial is determined by its leading term. Applying this to the numerator and denominator of $f(x)$, we get that as $x \to \pm\infty$, $f(x) = \frac{2x-1}{x+1} \approx \frac{2x}{x} = 2$. This last approach is useful in Calculus, and, indeed, is made rigorous there. (Keep this in mind for the remainder of this paragraph.) Applying this reasoning to the general case, suppose $r(x) = \frac{p(x)}{q(x)}$ where $a$ is the leading coefficient of $p(x)$ and $b$ is the leading coefficient of $q(x)$. As $x \to \pm\infty$, $r(x) \approx \frac{ax^n}{bx^m}$, where $n$ and $m$ are the degrees of $p(x)$ and $q(x)$, respectively. If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case. If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{a}{bx^{m-n}} \to 0$ as $x \to \pm\infty$. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive

[7] Note that as $x \to -\infty$, $f(x) \to 2^+$, whereas as $x \to \infty$, $f(x) \to 2^-$. We write $f(x) \to 2$ if we are unconcerned from which direction the function values $f(x)$ approach the number 2.

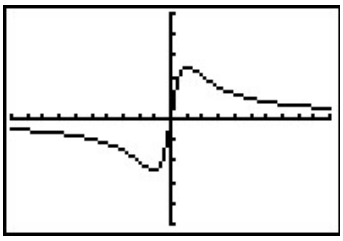
number and $r(x) \approx \frac{ax^{n-m}}{b}$, which becomes unbounded as $x \to \pm\infty$. As we said before, if a rational function has a horizontal asymptote, then it will have only one. (This is not true for other types of functions we shall see in later chapters.)

Example 5.1.4. Determine the horizontal asymptotes, if any, of the graphs of the following functions. Verify your answers using a graphing calculator.

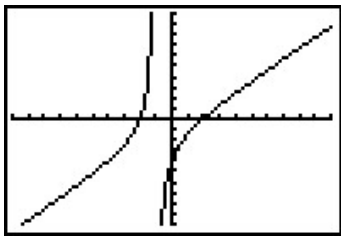
1. $f(x) = \dfrac{5x}{x^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(x) = \dfrac{6x^3-3x+1}{5-2x^3}$

Solution.

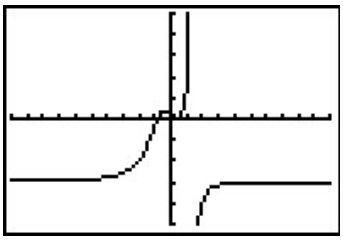
1. The numerator of $f(x)$ is $5x$, which is degree 1. The denominator of $f(x)$ is $x^2+1$, which is degree 2. Applying Theorem 5.2, $y = 0$ is the horizontal asymptote. Sure enough, as $x \to \pm\infty$, the graph of $y = f(x)$ gets closer and closer to the $x$-axis.
2. The numerator of $g(x)$, $x^2-4$, is degree 2, but the degree of the denominator, $x+1$, is degree 1. By Theorem 5.2, there is no horizontal asymptote. From the graph, we see the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote.[8]
3. The degrees of the numerator and denominator of $h(x)$ are both three, so Theorem 5.2 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. The calculator confirms this.



The graph of $y = f(x)$



The graph of $y = g(x)$



The graph of $y = h(x)$

□

Our last example of the section gives us a real-world application of a horizontal asymptote. Though the population below is more accurately modeled with the functions in Chapter 7, we approximate it[9] using a rational function.

Example 5.1.5. The number of students, $N$, at local college who have had the flu $t$ months after the semester begins can be modeled by the formula $N(t) = 500 - \frac{450}{1+3t}$ for $t \geq 0$.

1. Find and interpret $N(0)$.

[8] The graph does, however, seem to resemble a non-constant line as $x \to \pm\infty$. We will discuss this phenomenon in the next section.

[9] Using techniques you'll see in Calculus.

2. How long will it take until 300 students will have had the flu?

3. Determine the behavior of $N$ as $t \to \infty$. Interpret this result graphically and within the context of the problem.

SOLUTION.

1. $N(0) = 500 - \frac{450}{1+3(0)} = 50$. This means that at the beginning of the semester, 50 students have had the flu.

2. We set $N(t) = 300$ to get $500 - \frac{450}{1+3t} = 300$ and solve. Isolating the fraction gives $\frac{450}{1+3t} = 200$. Clearing denominators gives $450 = 200(1 + 3t)$. Finally, we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.

3. To determine the behavior of $N$ as $t \to \infty$, we can use a table.

| $t$ | $N(t)$ |
|---|---|
| 10 | $\approx 485.48$ |
| 100 | $\approx 498.50$ |
| 1000 | $\approx 499.85$ |
| 10000 | $\approx 499.98$ |

The table suggests that as $t \to \infty$, $N(t) \to 500$. (More specifically, $500^-$.) This means as time goes by, only a total of 500 students will have ever had the flu. □

### 5.1.1 Exercises

1. For each rational function $f$ given below:

   - Find the domain of $f$.
   - Identify any vertical asymptotes of the graph of $y = f(x)$ and describe the behavior of the graph near them using proper notation.
   - Identify any holes in the graph.
   - Find the horizontal asymptote, if it exists, and describe the end behavior of $f$ using proper notation.

   (a) $f(x) = \dfrac{x}{3x-6}$

   (b) $f(x) = \dfrac{3+7x}{5-2x}$

   (c) $f(x) = \dfrac{x}{x^2+x-12}$

   (d) $f(x) = \dfrac{x}{x^2+1}$

   (e) $f(x) = \dfrac{x+7}{(x+3)^2}$

   (f) $f(x) = \dfrac{x^3+1}{x^2-1}$

   (g) $f(x) = \dfrac{4x}{x^2+4}$

   (h) $f(x) = \dfrac{4x}{x^2-4}$

   (i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6}$

   (j) $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

   (k) $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

2. In Exercise 11 in Section 2.2, the population of Sasquatch in Portage County was modeled by the function $P(t) = \frac{150t}{t+15}$, where $t = 0$ represents the year 1803. Find the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

3. In Exercise 8 in Section 4.1, we fit a few polynomial models to the following electric circuit data. (The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, $mW$) is given in the table below.)[10]

| Resistance: $(k\Omega)$ | 1.012 | 2.199 | 3.275 | 4.676 | 6.805 | 9.975 |
|---|---|---|---|---|---|---|
| Power: $(mW)$ | 1.063 | 1.496 | 1.610 | 1.613 | 1.505 | 1.314 |

   Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power to resistance. For this circuit, it is $P(x) = \frac{25x}{(x+3.9)^2}$, where $x$ is the resistance value, $x \geq 0$.

   (a) Graph the data along with the function $y = P(x)$ on your calculator.

   (b) Approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

   (c) Find and interpret the end behavior of $P(x)$ as $x \to \infty$.

[10] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

4. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed. (This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.) Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where $L$ is the predicted practice limit in terms of speed units, $X$ is pages written, $Y$ is writing speed in terms of words in four minutes, $P$ is equivalent previous practice in terms of pages and $R$ is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

### 5.1.2 Answers

1. (a) $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty, f(x) \to \frac{1}{3}^-$
As $x \to \infty, f(x) \to \frac{1}{3}^+$

(b) $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-, f(x) \to \infty$
As $x \to \frac{5}{2}^+, f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty, f(x) \to -\frac{7}{2}^+$
As $x \to \infty, f(x) \to -\frac{7}{2}^-$

(c) $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4, x = 3$
As $x \to -4^-, f(x) \to -\infty$
As $x \to -4^+, f(x) \to \infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

(d) $f(x) = \dfrac{x}{x^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

(e) $f(x) = \dfrac{x+7}{(x+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $x = -3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[11]As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

(f) $f(x) = \dfrac{x^3+1}{x^2-1} = \dfrac{x^2-x+1}{x-1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $x = 1$
As $x \to 1^-, f(x) \to -\infty$
As $x \to 1^+, f(x) \to \infty$
Hole at $(-1, -\frac{3}{2})$
No horizontal asymptote
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$

(g) $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

[11]This is hard to see on the calculator, but trust me, the graph is below the $x$-axis to the left of $x = -7$.

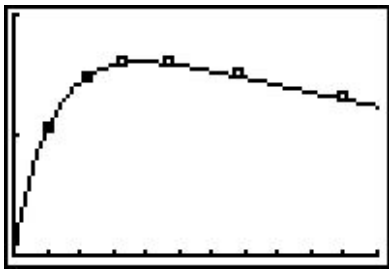
(h) $f(x) = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$
Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
Vertical asymptotes: $x=-2, x=2$
As $x\to -2^-, f(x)\to -\infty$
As $x\to -2^+, f(x)\to \infty$
As $x\to 2^-, f(x)\to -\infty$
As $x\to 2^+, f(x)\to \infty$
No holes in the graph
Horizontal asymptote: $y=0$
As $x\to -\infty, f(x)\to 0^-$
As $x\to \infty, f(x)\to 0^+$

(i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$
Domain: $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$
Vertical asymptote: $x=2$
As $x\to 2^-, f(x)\to \infty$
As $x\to 2^+, f(x)\to -\infty$
Hole at $\left(-3,\frac{7}{5}\right)$
Horizontal asymptote: $y=1$
As $x\to -\infty, f(x)\to 1^+$
As $x\to \infty, f(x)\to 1^-$

(j) $f(x) = \dfrac{3x^2-5x-2}{x^2-9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
Vertical asymptotes: $x=-3, x=3$
As $x\to -3^-, f(x)\to \infty$
As $x\to -3^+, f(x)\to -\infty$
As $x\to 3^-, f(x)\to -\infty$
As $x\to 3^+, f(x)\to \infty$
No holes in the graph
Horizontal asymptote: $y=3$
As $x\to -\infty, f(x)\to 3^+$
As $x\to \infty, f(x)\to 3^-$

(k) $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$
Vertical asymptote: $x=2$
As $x\to 2^-, f(x)\to -\infty$
As $x\to 2^+, f(x)\to -\infty$
Hole at $(-1,0)$
No horizontal asymptote
As $x\to -\infty, f(x)\to -\infty$
As $x\to \infty, f(x)\to \infty$

2. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y=150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

3. (a) 

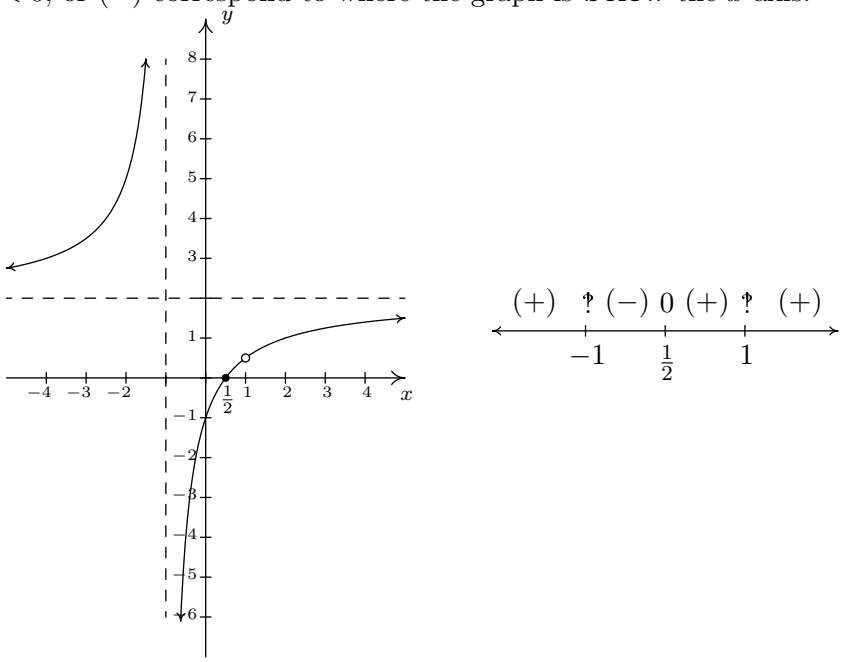
(b) The maximum power is approximately 1.603 $mW$ which corresponds to 3.9 $k\Omega$.

(c) As $x\to\infty$, $P(x)\to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

## 5.2 Graphs of Rational Functions

In this section, we take a closer look at graphing rational functions. In Section 5.1, we learned that the graphs of rational functions may include vertical asymptotes, holes in the graph, and horizontal asymptotes. Theorems 5.1 and 5.2 tell us exactly when and where these behaviors will occur, and if we combine these results with what we already know about graphing functions, we will quickly be able to generate reasonable graphs of rational functions.

One of the standard tools we will use is the sign diagram which was first introduced in Section 3.4, and then revisited in Section 4.1. In those sections, we operated under the belief that a function couldn't change its sign without its graph crossing through the $x$-axis. The major theorem we used to justify this belief was the Intermediate Value Theorem, Theorem 4.1. It turns out the Intermediate Value Theorem applies to all **continuous** functions,[1] not just polynomials. Although rational functions are continuous on their domains,[2] Theorem 5.1 tells us vertical asymptotes and holes occur at the values excluded from their domains. In other words, rational functions aren't continuous at these excluded values which leaves open the possibility that the function could change sign **without** crossing through the $x$-axis. Consider the graph of $y = h(x)$ from Example 5.1.1, recorded below for convenience. We have added its $x$-intercept at $\left(\frac{1}{2}, 0\right)$ for the discussion that follows. Suppose we wish to construct a sign diagram for $h(x)$. Recall that the intervals where $h(x) > 0$, or $(+)$, correspond to the $x$-values where the graph of $y = h(x)$ is **above** the $x$-axis; the intervals on which $h(x) < 0$, or $(-)$ correspond to where the graph is **below** the $x$-axis.




As we examine the graph of $y = h(x)$, reading from left to right, we note that from $(-\infty, -1)$,

[1] Recall that, for our purposes, this means the graphs are devoid of any breaks, jumps or holes

[2] Another result from Calculus.

the graph is above the $x$-axis, so $h(x)$ is $(+)$ there. At $x = -1$, we have a vertical asymptote, at which point the graph 'jumps' across the $x$-axis. On the interval $\left(-1, \frac{1}{2}\right)$, the graph is below the $x$-axis, so $h(x)$ is $(-)$ there. The graph crosses through the $x$-axis at $\left(\frac{1}{2}, 0\right)$ and remains above the $x$-axis until $x = 1$, where we have a 'hole' in the graph. Since $h(1)$ is undefined, there is no sign here. So we have $h(x)$ as $(+)$ on the interval $\left(\frac{1}{2}, 1\right)$. Continuing, we see that on $(1, \infty)$, the graph of $y = h(x)$ is above the $x$-axis, and so we mark $(+)$ there. To construct a sign diagram from this information, we not only need to denote the zero of $h$, but also the places not in the domain of $h$. As is our custom, we write '0' above $\frac{1}{2}$ on the sign diagram to remind us that it is a zero of $h$. We need a different notation for $-1$ and 1, and we have chosen to use '⁉' - a nonstandard symbol called the interrobang. We use this symbol to convey a sense of surprise, caution, and wonderment - an appropriate attitude to take when approaching these points. The moral of the story is that when constructing sign diagrams for rational functions, we include the zeros as well as the values excluded from the domain.

**Steps for Constructing a Sign Diagram for a Rational Function**

Suppose $r$ is a rational function.

1. Place any values excluded from the domain of $r$ on the number line with an '⁉' above them.
2. Find the zeros of $r$ and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $r(x)$ for each test value in step 3, and write that sign above the corresponding interval.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

**Steps for Graphing Rational Functions**

Suppose $r$ is a rational function.

1. Find the domain of $r$.
2. Reduce $r(x)$ to lowest terms, if applicable.
3. Find the $x$- and $y$-intercepts of the graph of $y = r(x)$, if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of $r$ on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of $r$. Use long division, as needed.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = r(x)$.

Example 5.2.1. Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2 - 4}$.

Solution. We follow the six step procedure outlined above.

1. As usual, we set the denominator equal to zero to get $x^2 - 4 = 0$. We find $x = \pm 2$, so our domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
2. To reduce $f(x)$ to lowest terms, we factor the numerator and denominator which yields $f(x) = \frac{3x}{(x-2)(x+2)}$. There are no common factors which means $f(x)$ is already in lowest terms.
3. To find the $x$-intercepts of the graph of $y = f(x)$, we set $y = f(x) = 0$. Solving $\frac{3x}{(x-2)(x+2)} = 0$ results in $x = 0$. Since $x = 0$ is in our domain, $(0, 0)$ is the $x$-intercept. To find the $y$-intercept, we set $x = 0$ and find $y = f(0) = 0$, so that $(0, 0)$ is our $y$-intercept as well.[3]
4. The two numbers excluded from the domain of $f$ are $x = -2$ and $x = 2$. Since $f(x)$ didn't reduce at all, both of these values of $x$ still cause trouble in the denominator, and so, by Theorem 5.1, $x = -2$ and $x = 2$ are vertical asymptotes of the graph. We can actually go a step farther at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,[4] it is good practice for those heading off to Calculus. For the discussion that follows, it is best to use the factored form of $f(x) = \frac{3x}{(x-2)(x+2)}$.

[3] As we mentioned at least once earlier, since functions can have at most one $y$-intercept, once we find $(0, 0)$ is on the graph, we know it is the $y$-intercept.

[4] The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

- *The behavior of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use $x$-values a little less than $-2$, say $-2.1$, $-2.01$ and $-2.001$. While there is no harm in actually building a table like we did in Section 5.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to $-6$, the quantity $(x-2)$ would be very close to $-4$, and the factor $(x+2)$ would be very close to 0. More specifically, $(x+2)$ would be a little less than 0, in this case, $-0.000001$. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value. So, mentally, as $x \to -2^-$, we estimate

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)\,(\text{very small } (-))} = \frac{3}{2\,(\text{very small } (-))}$$

  Now, the closer $x$ gets to $-2$, the smaller $(x+2)$ will become, and so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small } (-))} \approx \frac{3}{\text{very small } (-)} \approx \text{very big } (-)$$

  The term 'very big $(-)$' means a number with a large absolute value which is negative.[5] What all of this means is that as $x \to -2^-$, $f(x) \to -\infty$. Now suppose we wanted to determine the behavior of $f(x)$ as $x \to -2^+$. If we imagine substituting something a little larger than $-2$ in for $x$, say $-1.999999$, we mentally estimate

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small } (+))} = \frac{3}{2\,(\text{very small } (+))} \approx \frac{3}{\text{very small } (+)} \approx \text{very big } (+)$$

  We conclude that as $x \to -2^+$, $f(x) \to \infty$.

- *The behavior of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

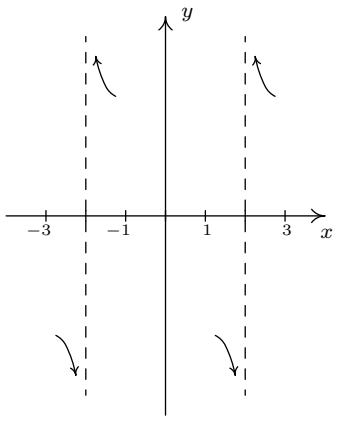
$$f(x) \approx \frac{6}{(\text{very small } (-))\,(4)} = \frac{3}{2\,(\text{very small } (-))} \approx \frac{3}{\text{very small } (-)} \approx \text{very big } (-)$$

  We conclude that as $x \to 2^-$, $f(x) \to -\infty$. Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$, we get $f(x) \approx \frac{3}{\text{very small } (+)} \approx$ very big $(+)$. So as $x \to 2^+$, $f(x) \to \infty$.

Graphically, we have that near $x = -2$ and $x = 2$ the graph of $y = f(x)$ looks like[6]

[5] The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.

[6] We have deliberately left off the labels on the $y$-axis because we know only the behavior near $x = \pm 2$, not the actual function values.




5. Next, we determine the end behavior of the graph of $y = f(x)$. Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 5.2 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \frac{3x}{x^2-4}$.

   - *The behavior of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behavior of $f$ as $x \to -\infty$, we would substitute very 'large' negative numbers in for $x$, say, for example, $x = -1$ billion. The numerator $3x$ would then be $-3$ billion, whereas the denominator $x^2 - 4$ would be $(-1 \text{ billion})^2 - 4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

   $$f(-1 \text{ billion}) \approx \frac{-3 \text{ billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small } (-)$$

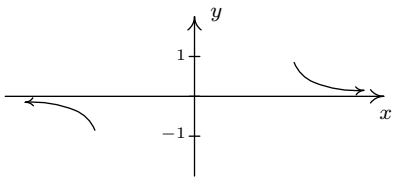
   Notice that if we substituted in $x = -1$ trillion, essentially the same kind of cancellation would happen, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit **below** the $x$-axis as we move to the far left.
   - *The behavior of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for $x$ and looking at the behavior of $f(x)$. For example, let $x = 1$ billion. Proceeding as before, we get

   $$f(1 \text{ billion}) \approx \frac{3 \text{ billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small } (+)$$

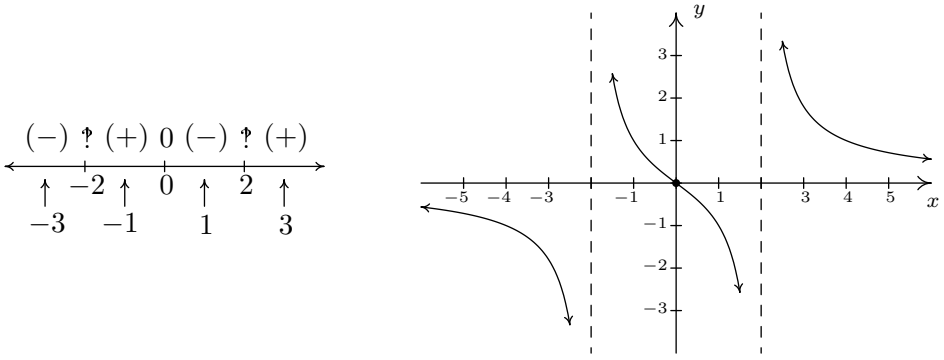
   The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit **above** the $x$-axis as we look toward the far right.

   Graphically, we have[7]

[7] As with the vertical asymptotes in the previous step, we know only the behavior of the graph as $x \to \pm\infty$. For that reason, we provide no $x$-axis labels.




6. Lastly, we construct a sign diagram for $f(x)$. The $x$-values excluded from the domain of $f$ are $x = \pm 2$, and the only zero of $f$ is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and we choose the test values[8] we $x = -3$, $x = -1$, $x = 1$, and $x = 3$. We find $f(-3)$ is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$, and $f(3)$ is $(+)$. Combining this with our previous work, we get the graph of $y = f(x)$ below.




□

A couple of notes are in order. First, the graph of $y = f(x)$ certainly seems to possess symmetry with respect to the origin. In fact, we can check $f(-x) = -f(x)$ to see that $f$ is an odd function. In some textbooks, checking for symmetry is part of the standard procedure for graphing rational functions; but since it happens comparatively rarely[9] we'll just point it out when we see it. Also note that while $y = 0$ is the horizontal asymptote, the graph of $f$ nevertheless crosses the $x$-axis at $(0, 0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false, as we shall see again in our next example.

EXAMPLE 5.2.2. Sketch a detailed graph of $g(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$.

SOLUTION.

1. Setting $x^2 - x - 6 = 0$ gives $x = -2$ and $x = 3$. Our domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.
2. Factoring $g(x)$ gives $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$. There is no cancellation, so $g(x)$ is in lowest terms.

[8] In this particular case, we can eschew test values, since our analysis of the behavior of $f$ near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.

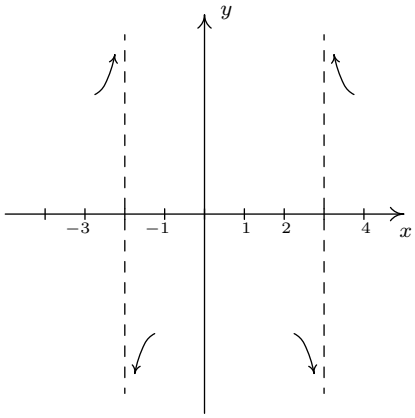
[9] And Jeff doesn't think much of it to begin with...

3. To find the $x$-intercept we set $y = g(x) = 0$. Using the factored form of $g(x)$ above, we find the zeros to be the solutions of $(2x-5)(x+1) = 0$. We obtain $x = \frac{5}{2}$ and $x = -1$. Since both of these numbers are in the domain of $g$, we have two $x$-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the $y$-intercept, we set $x = 0$ and find $y = g(0) = \frac{5}{6}$, so our $y$-intercept is $\left(0, \frac{5}{6}\right)$.

4. Since $g(x)$ was given to us in lowest terms, we have, once again by Theorem 5.1 vertical asymptotes $x = -2$ and $x = 3$. Keeping in mind $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$, we proceed to our analysis near each of these values.

   - *The behavior of $y = g(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than $-2$. We have
   $$g(x) \approx \frac{(-9)(-1)}{(-5)(\text{very small } (-))} \approx \frac{9}{\text{very small } (+)} \approx \text{very big } (+)$$
   so as $x \to -2^-$, $g(x) \to \infty$. On the flip side, as $x \to -2^+$, we get
   $$g(x) \approx \frac{9}{\text{very small } (-)} \approx \text{very big } (-)$$
   so $g(x) \to -\infty$.
   - *The behavior of $y = g(x)$ as $x \to 3$:* As $x \to 3^-$, we imagine plugging in a number just shy of 3. We have
   $$g(x) \approx \frac{(1)(4)}{(\text{ very small } (-))(5)} \approx \frac{4}{\text{very small } (-)} \approx \text{very big } (-)$$
   Hence, as $x \to 3^-$, $g(x) \to -\infty$. As $x \to 3^+$, we get
   $$g(x) \approx \frac{4}{\text{very small } (+)} \approx \text{very big } (+)$$
   so $g(x) \to \infty$.

   Graphically, we have (again, without labels on the $y$-axis)



5. Since the degrees of the numerator and denominator of $g(x)$ are the same, we know from Theorem 5.2 that we can find the horizontal asymptote of the graph of $g$ by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behavior which would be lost otherwise.[10] As in the discussion following Theorem 5.2, we use the result of the long division $\left(2x^2 - 3x - 5\right) \div \left(x^2 - x - 6\right)$ to rewrite $g(x) = \frac{2x^2-3x-5}{x^2-x-6}$ as $g(x) = 2 - \frac{x-7}{x^2-x-6}$. We focus our attention on the term $\frac{x-7}{x^2-x-6}$.

   - *The behavior of $y = g(x)$ as $x \to -\infty$:* If imagine substituting $x = -1$ billion into $\frac{x-7}{x^2-x-6}$, we estimate $\frac{x-7}{x^2-x-6} \approx \frac{-1 \text{ billion}}{1\text{billion}^2} \approx$ very small $(-)$.[11] Hence,

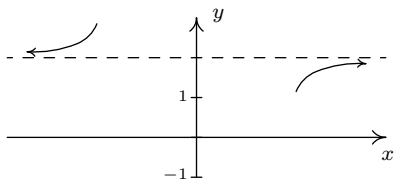
   $$g(x) = 2 - \frac{x-7}{x^2-x-6} \approx 2 - \text{very small } (-) = 2 + \text{very small } (+)$$

   In other words, as $x \to -\infty$, the graph of $y = g(x)$ is a little bit **above** the line $y = 2$.
   - *The behavior of $y = g(x)$ as $x \to \infty$.* To consider $\frac{x-7}{x^2-x-6}$ as $x \to \infty$, we imagine substituting $x = 1$ billion and, going through the usual mental routine, find

   $$\frac{x-7}{x^2-x-6} \approx \text{very small } (+)$$

   Hence, $g(x) \approx 2 -$ very small $(+)$, in other words, the graph of $y = g(x)$ is just **below** the line $y = 2$ as $x \to \infty$.

   On $y = g(x)$, we have (again, without labels on the $x$-axis)

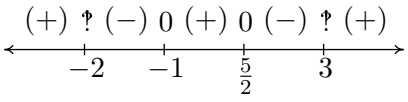


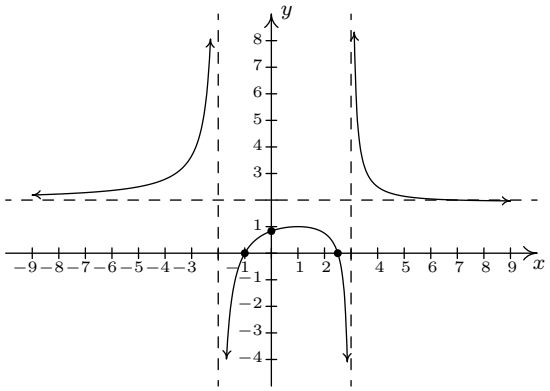

6. Finally we construct our sign diagram. We place an '‽' above $x = -2$ and $x = 3$, and a '0' above $x = \frac{5}{2}$ and $x = -1$. Choosing test values in the test intervals gives us $f(x)$ is $(+)$ on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$, and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$. As we piece together all of the information, we note that the graph must cross the horizontal asymptote at some point after $x = 3$ in order for it to approach $y = 2$ from underneath. This is the subtlety that we would have missed had we skipped the long division and subsequent end behavior analysis. We can, in fact, find exactly when the graph crosses $y = 2$. As a result

[10] That is, if you use a calculator to graph. Once again, Calculus is the ultimate graphing power tool.

[11] In the denominator, we would have $(1\text{billion})^2 - 1\text{billion} - 6$. It's easy to see why the 6 is insignificant, but to ignore the 1 billion seems criminal. However, compared to $(1 \text{ billion})^2$, it's on the insignificant side; it's $10^{18}$ versus $10^9$. We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the $x^2$ term wins out over the $x$ term.

of the long division, we have $g(x) = 2 - \frac{x-7}{x^2-x-6}$. For $g(x) = 2$, we would need $\frac{x-7}{x^2-x-6} = 0$. This gives $x - 7 = 0$, or $x = 7$. Note that $x - 7$ is the remainder when $2x^2 - 3x - 5$ is divided by $x^2 - x - 6$, and so it makes sense that for $g(x)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7) = 2$. Moreover, it stands to reason that $g$ must attain a relative minimum at some point past $x = 7$. Calculus verifies that at $x = 13$, we have such a minimum at exactly $(13, 1.96)$. The reader is challenged to find calculator windows which show the graph crossing its horizontal asymptote on one window, and the relative minimum in the other.




□

Our next example gives us not only a hole in the graph, but also some slightly different end behavior.

Example 5.2.3. Sketch a detailed graph of $h(x) = \dfrac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$.

Solution.

1. For domain, you know the drill. Solving $x^2 + 3x + 2 = 0$ gives $x = -2$ and $x = -1$. Our answer is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

2. To reduce $h(x)$, we need to factor the numerator and denominator. We get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} = \frac{(2x+1)(x+1)^{\cancel{2}^{1}}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

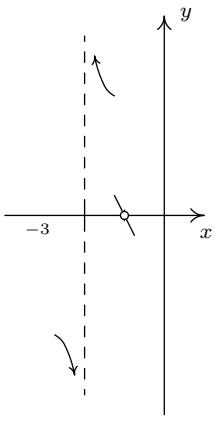
   We will use this reduced formula for $h(x)$ as long as we're not substituting $x = -1$. To make this exclusion specific, we write $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

3. To find the $x$-intercepts, as usual, we set $h(x) = 0$ and solve. Solving $\frac{(2x+1)(x+1)}{x+2} = 0$ yields $x = -\frac{1}{2}$ and $x = -1$. The latter isn't in the domain of $h$, so we exclude it. Our only $x$-intercept is $\left(-\frac{1}{2}, 0\right)$. To find the $y$-intercept, we set $x = 0$. Since $0 \neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0) = \frac{1}{2}$ for a $y$-intercept of $\left(0, \frac{1}{2}\right)$.

4. From Theorem 5.1, we know that since $x = -2$ still poses a threat in the denominator of the reduced function, we have a vertical asymptote there. As for $x = -1$, we note the factor $(x+1)$ was canceled from the denominator when we reduced $h(x)$, and so it no longer causes trouble there. This means we get a hole when $x = -1$. To find the $y$-coordinate of the hole, we substitute $x = -1$ into $\frac{(2x+1)(x+1)}{x+2}$, per Theorem 5.1 and get 0. Hence, we have a hole on the $x$-axis at $(-1, 0)$. It should make you uncomfortable plugging $x = -1$ into the reduced formula for $h(x)$, especially since we've made such a big deal concerning the stipulation about not letting $x = -1$ for that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near $x = -1$ which we will show below. Speaking of which, for the discussion that follows, we will use the formula $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

   - *The behavior of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than $-2$. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small } (-))} \approx \frac{3}{(\text{very small } (-))} \approx$ very big $(-)$ and so as $x \to -2^-$, $h(x) \to -\infty$. On the other side of $-2$, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small } (+)} \approx$ very big $(+)$, so $h(x) \to \infty$.
   - *The behavior of $y = h(x)$ as $x \to -1$.* As $x \to -1^-$, we imagine plugging in a number a bit less than $x = -1$. We have $h(x) \approx \frac{(-1)(\text{very small } (-))}{1} =$ very small $(+)$ Hence, as $x \to -1^-$, $h(x) \to 0^+$. This means, as $x \to -1^-$, the graph is a bit above the point $(-1, 0)$. As $x \to -1^+$, we get $h(x) \approx \frac{(-1)(\text{very small } (+))}{1} =$ very small $(-)$. This gives us that as $x \to -1^+$, $h(x) \to 0^-$, so the graph is a little bit lower than $(-1, 0)$ here.

   Graphically, we have

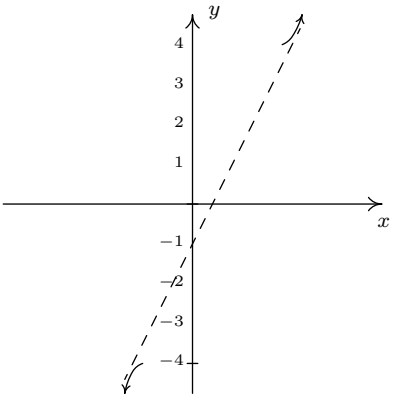



5. For end behavior, we note that the degree of the numerator of $h(x)$, $2x^3 + 5x^2 + 4x + 1$ is 3, and the degree of the denominator, $x^2 + 3x + 2$, is 2. Theorem 5.2 is of no help here, since the degree of the numerator is greater than the degree of the denominator. That won't stop us, however, in our analysis. Since for end behavior we are considering values of $x$ as $x \to \pm\infty$, we are far enough away from $x = -1$ to use the reduced formula, $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \frac{2x^2+3x+1}{x+2}$, $x \neq -1$,

and, as a result, we rewrite $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$. As in the previous example, we focus our attention on the term generated from the remainder, $\frac{3}{x+2}$.

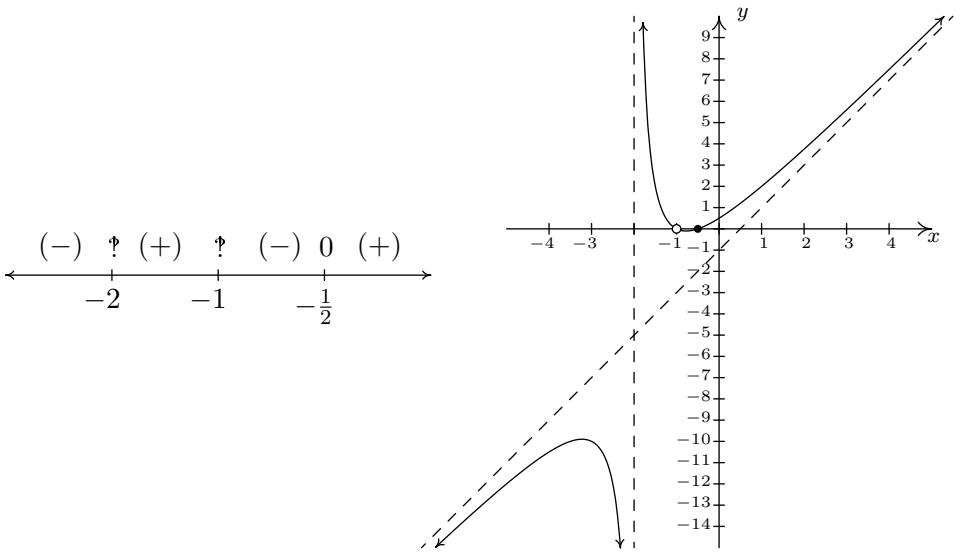
- *The behavior of $y = h(x)$ as $x \to -\infty$:* Substituting $x = -1$ billion into $\frac{3}{x+2}$, we get the estimate $\frac{3}{-1 \text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x - 1 + \frac{3}{x+2} \approx 2x - 1 +$ very small $(-)$. This means the graph of $y = h(x)$ is a little bit **below** the line $y = 2x - 1$ as $x \to -\infty$.
- *The behavior of $y = h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\frac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x - 1 +$ very small $(+)$, or that the graph of $y = h(x)$ is a little bit **above** the line $y = 2x - 1$ as $x \to \infty$.

This is end behavior unlike any we've ever seen. Instead of approaching a horizontal line, the graph is approaching a slanted line. For this reason, $y = 2x - 1$ is called a **slant asymptote**[12] of the graph of $y = h(x)$. A slant asymptote will always arise when the degree of the numerator is exactly one more than the degree of the denominator, and there's no way to determine exactly what it is without going through the long division. Graphically we have




6. To make our sign diagram, we place an '‖' above $x = -2$ and $x = -1$ and a '0' above $x = -\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2, -1)$ and $\left(-\frac{1}{2}, \infty\right)$ and $h(x)$ is $(-)$ on $(-\infty, -2)$ and $\left(-1, -\frac{1}{2}\right)$. Putting all of our work together yields the graph below.

[12] Also called an 'oblique' asymptote in some texts.




We could ask whether the graph of $y = h(x)$ crosses its slant asymptote. From the formula $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$, we see that if $h(x) = 2x - 1$, we would have $\frac{3}{x+2} = 0$. Since this will never happen, we conclude the graph never crosses its slant asymptote.[13] □

We end this section with an example that shows it's not all pathological weirdness when it comes to rational functions and technology still has a role to play in studying their graphs at this level.

EXAMPLE 5.2.4. Sketch the graph of $r(x) = \dfrac{x^4 + 1}{x^2 + 1}$.

SOLUTION.

1. The denominator $x^2 + 1$ is never zero so the domain is $(-\infty, \infty)$.

2. With no real zeros in the denominator, $x^2 + 1$ is an irreducible quadratic. Our only hope of reducing $r(x)$ is if $x^2 + 1$ is a factor of $x^4 + 1$. Performing long division gives us

$$\frac{x^4 + 1}{x^2 + 1} = x^2 - 1 + \frac{2}{x^2 + 1}$$

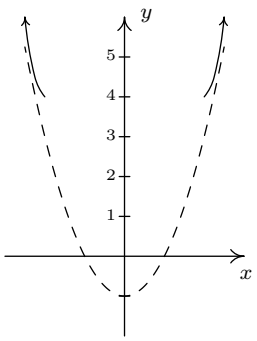
   The remainder is not zero so $r(x)$ is already reduced.

3. To find the $x$-intercept, we'd set $r(x) = 0$. Since there are no real solutions to $\frac{x^4+1}{x^2+1} = 0$, we have no $x$-intercepts. Since $r(0) = 1$, so we get $(0, 1)$ for the $y$-intercept.

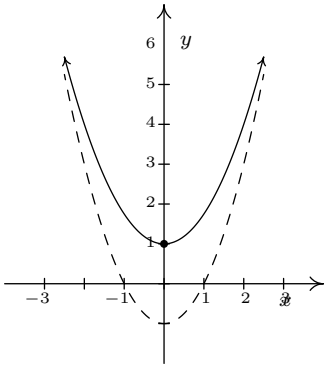
4. This step doesn't apply to $r$, since its domain is all real numbers.

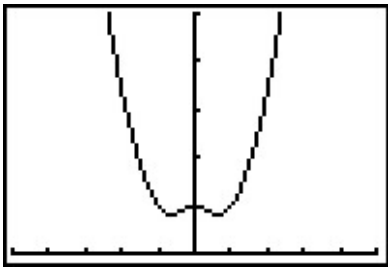
[13] But rest assured, some graphs do!

5. For end behavior, once again, since the degree of the numerator is greater than that of the denominator, Theorem 5.2 doesn't apply. We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \frac{2}{x^2+1}$, and so we focus our attention on the term corresponding to the remainder, $\frac{2}{x^2+1}$ It should be clear that as $x \to \pm\infty$, $\frac{2}{x^2+1} \approx$ very small $(+)$, which means $r(x) \approx x^2 - 1 +$ very small $(+)$. So the graph $y = r(x)$ is a little bit **above** the graph of the parabola $y = x^2 - 1$ as $x \to \pm\infty$. Graphically,




6. There isn't much work to do for a sign diagram for $r(x)$, since its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty, \infty)$, and since we know $r(0) = 1$, we conclude $r(x)$ is $(+)$ for all real numbers. At this point, we don't have much to go on for a graph. Below is a comparison of what we have determined analytically versus what the calculator shows us. We have no way to detect the relative extrema analytically[14] apart from brute force plotting of points, which is done more efficiently by the calculator.



□

[14] Without appealing to Calculus, of course.

### 5.2.1 EXERCISES

1. Find the slant asymptote of the graph of the rational function.

   (a) $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$

   (b) $f(x) = \dfrac{2x^2 + 5x - 3}{3x + 2}$

   (c) $f(x) = \dfrac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$

   (d) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

2. Use the six-step procedure to graph each rational function given. Be sure to draw any asymptotes as dashed lines.

   (a) $f(x) = \dfrac{4}{x + 2}$

   (b) $f(x) = \dfrac{5x}{6 - 2x}$

   (c) $f(x) = \dfrac{1}{x^2}$

   (d) $f(x) = \dfrac{1}{x^2 + x - 12}$

   (e) $f(x) = \dfrac{2x - 1}{-2x^2 - 5x + 3}$

   (f) $f(x) = \dfrac{x}{x^2 + x - 12}$

   (g) $f(x) = \dfrac{4x}{x^2 + 4}$

   (h) $f(x) = \dfrac{4x}{x^2 - 4}$

   (i) $f(x) = \dfrac{x^2 - x - 12}{x^2 + x - 6}$

   (j) $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9}$

   (k) $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$

   (l) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

   (m) [15] $f(x) = \dfrac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

3. Example 5.2.4 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function. Without Calculus, we need to use our graphing calculators to reveal the hidden mysteries of rational function behavior. Working with your classmates, use a graphing calculator to examine the graphs of the following rational functions. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

   (a) $f(x) = \dfrac{1}{x^2 + 1}$

   (b) $f(x) = \dfrac{x}{x^2 + 1}$

   (c) $f(x) = \dfrac{x^2}{x^2 + 1}$

   (d) $f(x) = \dfrac{x^3}{x^2 + 1}$

[15] Once you've done the six-step procedure, use your calculator to graph this function on the viewing window $[0, 12] \times [0, 0.25]$. What do you see?

4. Graph the following rational functions by applying transformations to the graph of $y = \frac{1}{x}$.

   (a) $f(x) = \dfrac{1}{x-2}$

   (b) $g(x) = 1 - \dfrac{3}{x}$

   (c) $h(x) = \dfrac{-2x+1}{x}$ (Hint: Divide)

   (d) $j(x) = \dfrac{3x-7}{x-2}$ (Hint: Long division)

   Discuss with your classmates how you would graph $f(x) = \dfrac{ax+b}{cx+d}$. What restrictions must be placed on $a, b, c$ and $d$ so that the graph is indeed a transformation of $y = \dfrac{1}{x}$?

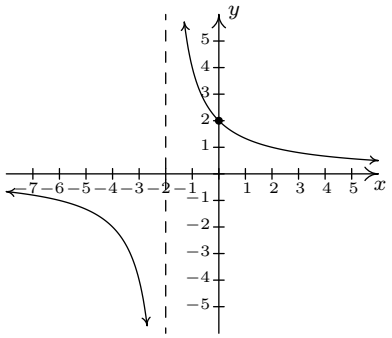
5. In Example 4.1.1 in Section 4.1 we showed that $p(x) = \frac{4x+x^3}{x}$ is not a polynomial even though its formula reduced to $4 + x^2$ for $x \neq 0$. However, it is a rational function similar to those studied in the section. With the help of your classmates, graph $p(x)$.

6. Let $g(x) = \dfrac{x^4 - 8x^3 + 24x^2 - 72x + 135}{x^3 - 9x^2 + 15x - 7}$. With the help of your classmates, find the $x$- and $y$- intercepts of the graph of $g$. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Find all of the asymptotes of the graph of $g$ and any holes in the graph, if they exist. Be sure to show all of your work including any polynomial or synthetic division. Sketch the graph of $g$, using more than one picture if necessary to show all of the important features of the graph.

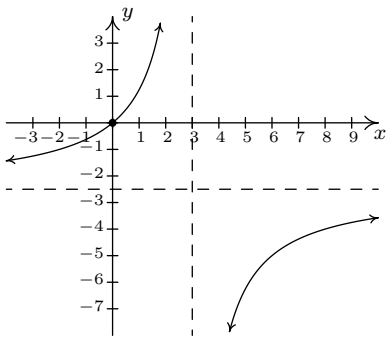
### 5.2.2 ANSWERS

1. (a) $y = x$
   (b) $y = \frac{2}{3}x + \frac{11}{9}$
   (c) $y = -5x - 18$
   (d) $y = -x$

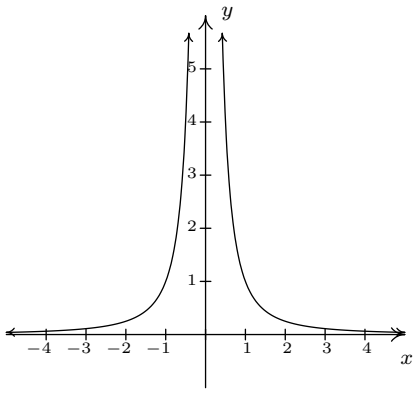
2. (a) $f(x) = \dfrac{4}{x+2}$
   Domain: $(-\infty, -2) \cup (-2, \infty)$
   No $x$-intercepts
   $y$-intercept: $(0, 2)$
   Vertical asymptote: $x = -2$
   As $x \to -2^-,\ f(x) \to -\infty$
   As $x \to -2^+,\ f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty,\ f(x) \to 0^-$
   As $x \to \infty,\ f(x) \to 0^+$




   (b) $f(x) = \dfrac{5x}{6-2x}$
   Domain: $(-\infty, 3) \cup (3, \infty)$
   $x$-intercept: $(0, 0)$
   $y$-intercept: $(0, 0)$
   Vertical asymptote: $x = 3$
   As $x \to 3^-,\ f(x) \to \infty$
   As $x \to 3^+,\ f(x) \to -\infty$
   Horizontal asymptote: $y = -\frac{5}{2}$
   As $x \to -\infty,\ f(x) \to -\frac{5}{2}^+$
   As $x \to \infty,\ f(x) \to -\frac{5}{2}^-$

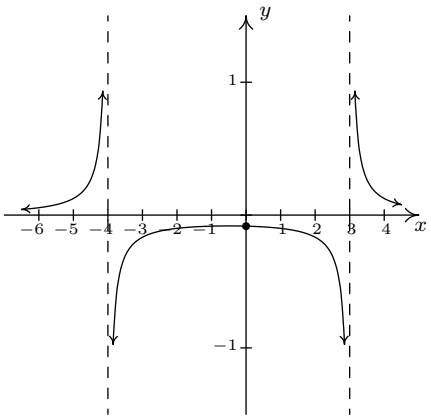



   (c) $f(x) = \dfrac{1}{x^2}$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   No $x$-intercepts
   No $y$-intercepts
   Vertical asymptote: $x = 0$
   As $x \to 0^-,\ f(x) \to \infty$
   As $x \to 0^+,\ f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty,\ f(x) \to 0^+$
   As $x \to \infty,\ f(x) \to 0^+$



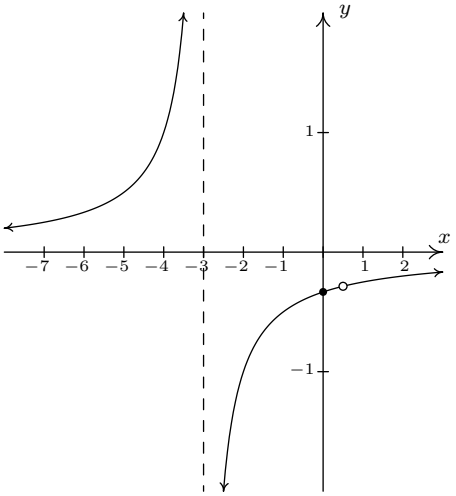
(d) $f(x) = \dfrac{1}{x^2 + x - 12}$

Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
No $x$-intercepts
$y$-intercept: $(0, -\frac{1}{12})$
Vertical asymptotes: $x = -4$ and $x = 3$
As $x \to -4^-,\ f(x) \to \infty$
As $x \to -4^+,\ f(x) \to -\infty$
As $x \to 3^-,\ f(x) \to -\infty$
As $x \to 3^+,\ f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty,\ f(x) \to 0^+$
As $x \to \infty,\ f(x) \to 0^+$



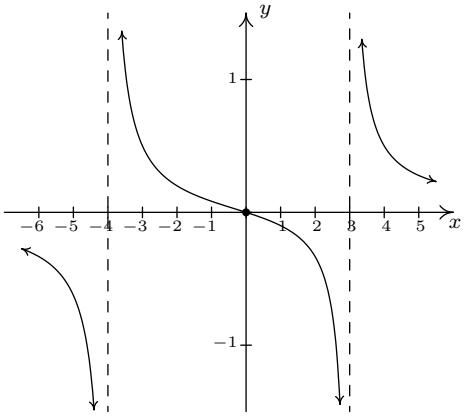

(e) $f(x) = \dfrac{2x - 1}{-2x^2 - 5x + 3}$

Domain: $(-\infty, -3) \cup (-3, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$
No $x$-intercepts
$y$-intercept: $(0, -\frac{1}{3})$
$f(x) = \dfrac{-1}{x + 3},\ x \neq \frac{1}{2}$
Hole in the graph at $(\frac{1}{2}, -\frac{2}{7})$
Vertical asymptote: $x = -3$
As $x \to -3^-,\ f(x) \to \infty$
As $x \to -3^+,\ f(x) \to -\infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty,\ f(x) \to 0^+$
As $x \to \infty,\ f(x) \to 0^+$



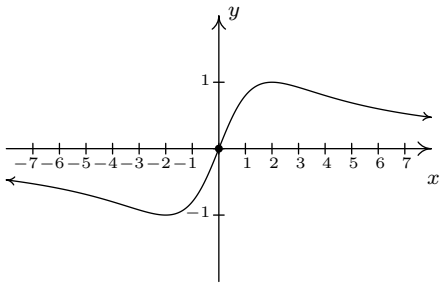

(f) $f(x) = \dfrac{x}{x^2 + x - 12}$

Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
$x$-intercept: $(0, 0)$
$y$-intercept: $(0, 0)$
Vertical asymptotes: $x = -4$ and $x = 3$
As $x \to -4^-,\ f(x) \to -\infty$
As $x \to -4^+,\ f(x) \to \infty$
As $x \to 3^-,\ f(x) \to -\infty$
As $x \to 3^+,\ f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty,\ f(x) \to 0^-$
As $x \to \infty,\ f(x) \to 0^+$



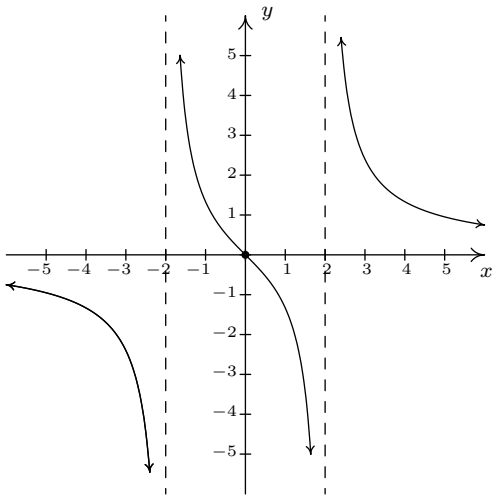
(g) $f(x) = \dfrac{4x}{x^2+4}$

Domain: $(-\infty, \infty)$
$x$-intercept: $(0,0)$
$y$-intercept: $(0,0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



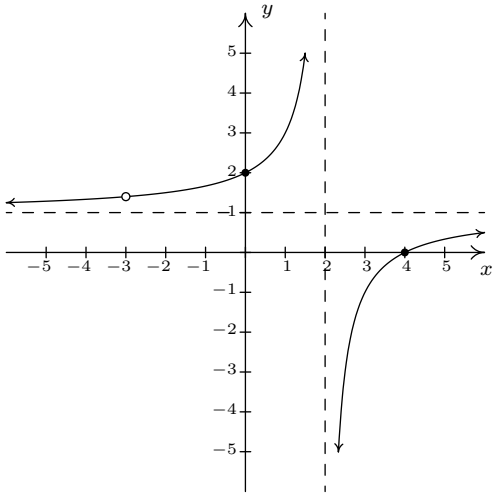

(h) $f(x) = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
$x$-intercept: $(0,0)$
$y$-intercept: $(0,0)$
Vertical asymptotes: $x = -2, x = 2$
As $x \to -2^-, f(x) \to -\infty$
As $x \to -2^+, f(x) \to \infty$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



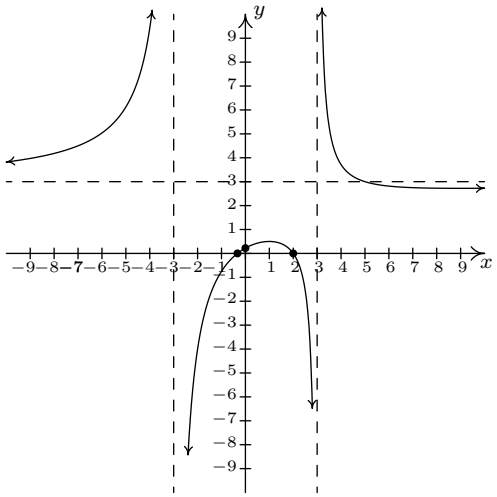

(i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$

Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
$x$-intercept: $(4,0)$
$y$-intercept: $(0,2)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$



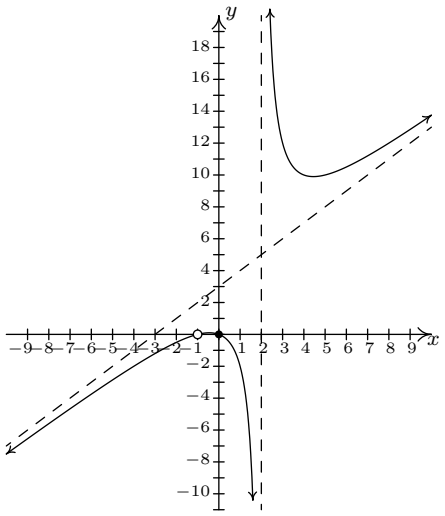
(j) $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
$x$-intercepts: $\left(-\frac{1}{3}, 0\right)$, $(2, 0)$
$y$-intercept: $\left(0, \frac{2}{9}\right)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to -\infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty, f(x) \to 3^+$
As $x \to \infty, f(x) \to 3^-$



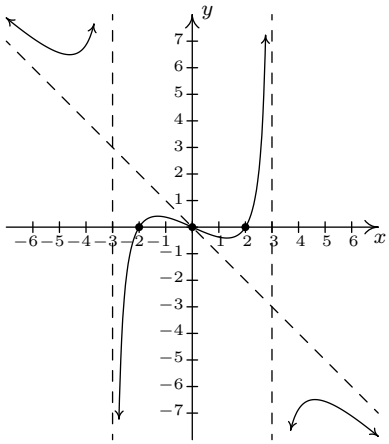
(k) $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$

Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
$x$-intercept: $(0, 0)$
$y$-intercept: $(0, 0)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$



(l) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
$x$-intercepts: $(-2, 0), (0, 0), (2, 0)$
$y$-intercept: $(0, 0)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-,\ f(x) \to \infty$
As $x \to -3^+,\ f(x) \to -\infty$
As $x \to 3^-,\ f(x) \to \infty$
As $x \to 3^+,\ f(x) \to -\infty$
Slant asymptote: $y = -x$
As $x \to -\infty,\ f(x) \to \infty$
As $x \to \infty,\ f(x) \to -\infty$



(m) $f(x) = \dfrac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)$

$f(x) = \dfrac{x - 1}{x(x + 2)},\ x \neq 1$

No $x$-intercepts

No $y$-intercepts

Vertical asymptotes: $x = -2$ and $x = 0$

As $x \to -2^-,\ f(x) \to -\infty$

As $x \to -2^+,\ f(x) \to \infty$

As $x \to 0^-,\ f(x) \to \infty$

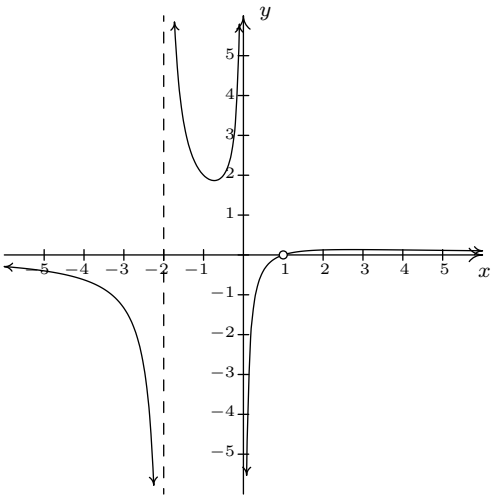
As $x \to 0^+,\ f(x) \to -\infty$

Hole in the graph at $(1, 0)$

Horizontal asymptote: $y = 0$

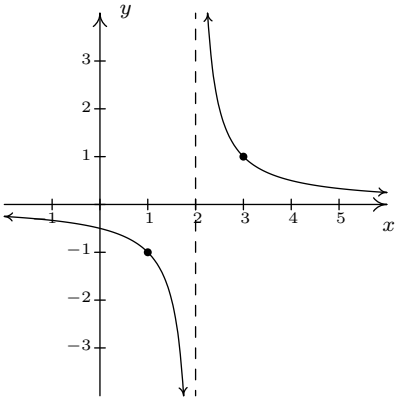
As $x \to -\infty,\ f(x) \to 0^-$

As $x \to \infty,\ f(x) \to 0^+$




4. (a) $f(x) = \dfrac{1}{x - 2}$

Shift the graph of $y = \dfrac{1}{x}$ to the right 2 units.

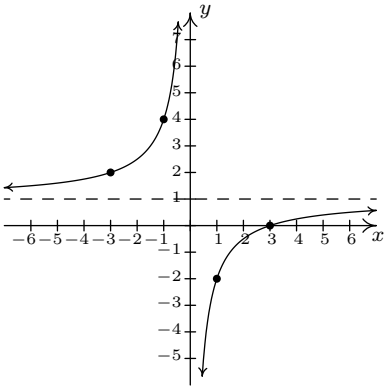



(b) $g(x) = 1 - \dfrac{3}{x}$

Vertically stretch the graph of $y = \dfrac{1}{x}$ by a factor of 3.

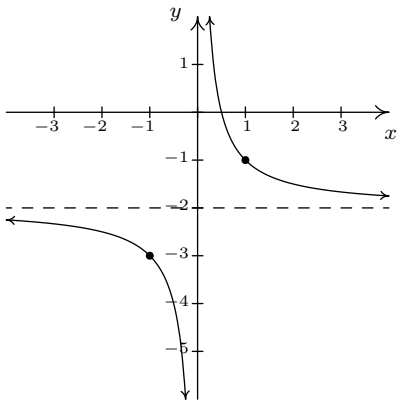
Reflect the graph of $y = \dfrac{3}{x}$ about the $x$-axis.

Shift the graph of $y = -\dfrac{3}{x}$ up 1 unit.



(c) $h(x) = \dfrac{-2x+1}{x} = -2 + \dfrac{1}{x}$

Shift the graph of $y = \dfrac{1}{x}$ down 2 units.

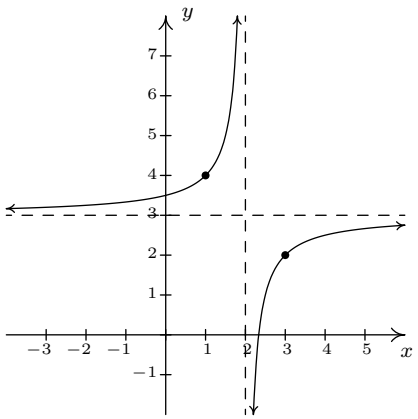



(d) $j(x) = \dfrac{3x-7}{x-2} = 3 - \dfrac{1}{x-2}$

Shift the graph of $y = \dfrac{1}{x}$ to the right 2 units.

Reflect the graph of $y = \dfrac{1}{x-2}$ about the $x$-axis.

Shift the graph of $y = -\dfrac{1}{x-2}$ up 3 units.



## 5.3 RATIONAL INEQUALITIES AND APPLICATIONS

In this section, we use sign diagrams to solve rational inequalities including some that arise from real-world applications. Our first example showcases the critical difference in procedure between solving a rational equation and a rational inequality.

EXAMPLE 5.3.1.

1. Solve $\dfrac{x^3-2x+1}{x-1}=\dfrac{1}{2}x-1$.
2. Solve $\dfrac{x^3-2x+1}{x-1}\geq\dfrac{1}{2}x-1$.
3. Use your calculator to graphically check your answers to 1 and 2.

SOLUTION.

1. To solve the equation, we clear denominators

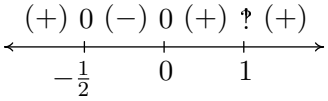
$$\begin{aligned}
\frac{x^3-2x+1}{x-1} &= \frac{1}{2}x-1 \\
\left(\frac{x^3-2x+1}{x-1}\right)\cdot 2(x-1) &= \left(\frac{1}{2}x-1\right)\cdot 2(x-1) \\
2x^3-4x+2 &= x^2-3x+2 && \text{expand} \\
2x^3-x^2-x &= 0 \\
x(2x+1)(x-1) &= 0 && \text{factor} \\
x &= -\tfrac{1}{2},\ 0,\ 1
\end{aligned}$$

   Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x=1$ does not satisfy the original equation and must be discarded. Our solutions are $x=-\frac{1}{2}$ and $x=0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation – namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on $x$, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 264 in Section 5.2.

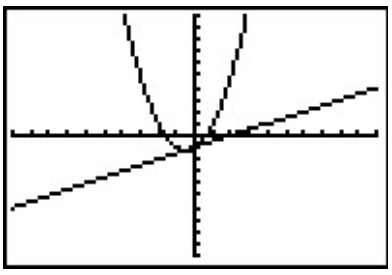
$$\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1}-\frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right)-x(x-1)+1(2(x-1))}{2(x-1)} &\geq 0 && \text{get a common denominator} \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 && \text{expand}
\end{aligned}$$

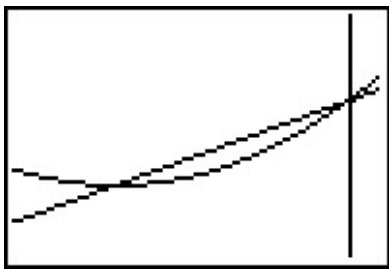
Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of $r$ is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of $r$ are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.




We are interested in where $r(x) \geq 0$. We find $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0, 1)$ and $(1, \infty)$. We add to these intervals the zeros of $r$, $-\frac{1}{2}$ and 0, to get our final solution: $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. Geometrically, if we set $f(x) = \frac{x^3-2x+1}{x-1}$ and $g(x) = \frac{1}{2}x - 1$, the solutions to $f(x) = g(x)$ are the $x$-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above $(>)$ the graph of $g(x)$. We obtain the graphs below.





The 'Intersect' command confirms that the graphs cross when $x = -\frac{1}{2}$ and $x = 0$. It is clear from the calculator that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $\left(-\infty, -\frac{1}{2}\right)$ as well as on $(0, \infty)$. According to the calculator, our solution is then $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$

which **almost** matches the answer we found analytically. We have to remember that $f$ is not defined at $x = 1$, and, even though it isn't shown on the calculator, there is a hole[1] in the graph of $y = f(x)$ when $x = 1$ which is why $x = 1$ needs to be excluded from our final answer. □

Our next example deals with the **average cost** function of PortaBoy Game systems from Example 3.1.5 in Section 3.1.

EXAMPLE 5.3.2. Given a cost function $C(x)$, which returns the total cost of producing $x$ products, the **average cost** function, $AC(x) = \frac{C(x)}{x}$, computes the cost per item. Recall that the cost $C$, in dollars, to produce $x$ PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function $AC(x)$. Determine an appropriate applied domain for $AC$.
2. Find and interpret $AC(10)$.
3. Solve $AC(x) < 100$ and interpret.
4. Determine the behavior of $AC(x)$ as $x \to \infty$ and interpret.

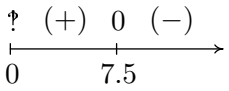
SOLUTION.

1. From $AC(x) = \frac{C(x)}{x}$, we obtain $AC(x) = \frac{80x+150}{x}$. The domain of $C$ is $x \geq 0$, but since $x = 0$ causes problems for $AC(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.
2. We find $AC(10) = \frac{80(10)+150}{10} = 95$, so the average cost to produce 10 game systems is \$95 per system.
3. Solving $AC(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x + 150}{x} &< 100 \\ \frac{80x + 150}{x} - 100 &< 0 \\ \frac{80x + 150 - 100x}{x} &< 0 \qquad \text{common denominator} \\ \frac{150 - 20x}{x} &< 0 \end{aligned}$$

If we take the left hand side to be a rational function $r(x)$, we need to keep in mind the the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, $r$ is defined everywhere so we need only look for zeros of $r$. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.

[1] There is no asymptote at $x = 1$ since the graph is well behaved near $x = 1$. According to Theorem 5.1, there must be a hole there.

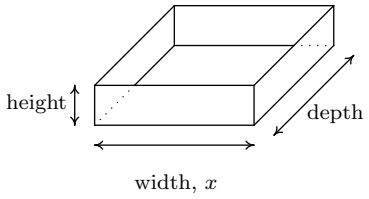



In the context of the problem, $x$ represents the number of PortaBoy games systems produced and $AC(x)$ is the average cost to produce each system. Solving $AC(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

4. We can apply Theorem 5.2 to $AC(x)$ and we find $y = 80$ is a horizontal asymptote to the graph of $y = AC(x)$. To more precisely determine the behavior of $AC(x)$ as $x \to \infty$, we first use long division[2] and rewrite $AC(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^+$, which means $AC(x) \approx 80 +$ very small $(+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $AC(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $AC(x) > 80$ for all $x > 0$. This means the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 3.1.5, we realize \$80 is the variable cost per system − the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively counterbalance the fixed cost. □

Our next example is another classic 'box with no top' problem.

Example 5.3.3. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let $x$ denote the width of the box, in centimeters. Refer to the figure below.




1. Express the height $h$ in centimeters as a function of the width $x$ and state the applied domain.
2. Solve $h(x) \geq x$ and interpret.
3. Find and interpret the behavior of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.
4. Express the surface area $S$ of the box as a function of $x$ and state the applied domain.
5. Use a calculator to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

[2] In this case, long division amounts to term-by-term division.

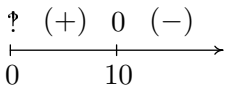
SOLUTION.

1. We are told the volume of the box is 1000 cubic centimeters and that $x$ represents the width, in centimeters. From geometry, we know Volume = width $\times$ height $\times$ depth. Since the base of the box is to be a square, the width and the depth are both $x$ centimeters. Using $h$ for the height, we have $1000 = x^2 h$, so that $h = \frac{1000}{x^2}$. Using function notation,[3] $h(x) = \frac{1000}{x^2}$ As for the applied domain, in order for there to be a box at all, $x > 0$, and since every such choice of $x$ will return a positive number for the height $h$ we have no other restrictions and conclude our domain is $(0, \infty)$.

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality and use a sign diagram.

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator} \end{aligned}$$

   We consider the left hand side of the inequality as our rational function $r(x)$. We see $r$ is undefined at $x = 0$, but, as in the previous example, the applied domain of the problem is $x > 0$, so we are considering only the behavior of $r$ on $(0, \infty)$. The sole zero of $r$ comes when $1000 - x^3 = 0$, which is $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the following diagram.



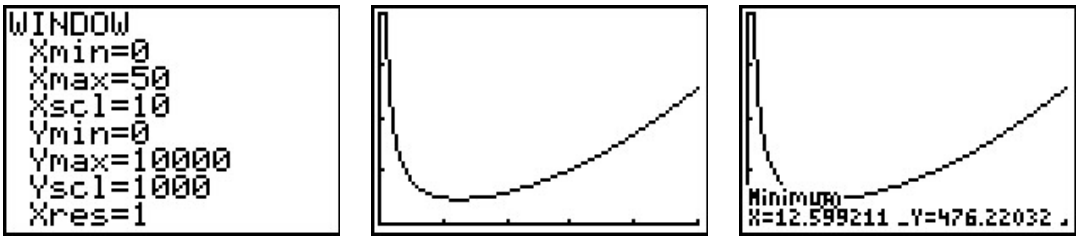

   We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, $h$ represents the height of the box while $x$ represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of $x$ which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.

3. As $x \to 0^+$, $h(x) = \frac{1000}{x^2} \to \infty$. This means the smaller the width $x$ (and, in this case, depth), the larger the height $h$ has to be in order to maintain a volume of 1000 cubic centimeters. As $x \to \infty$, we find $h(x) \to 0^+$, which means to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger the smaller the height becomes.

[3] That is, $h(x)$ means '$h$ of $x$', not '$h$ times $x$' here.

4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions $x$ by $x$, and each side has dimensions $x$ by $h$. We get the surface area, $S = x^2 + 4xh$. To get $S$ as a function of $x$, we substitute $h = \frac{1000}{x^2}$ to obtain $S = x^2 + 4x\left(\frac{1000}{x^2}\right)$. Hence, as a function of $x$, $S(x) = x^2 + \frac{4000}{x}$. The domain of $S$ is the same as $h$, namely $(0, \infty)$, for the same reasons as above.

5. A first attempt at the graph of $y = S(x)$ on the calculator may lead to frustration. Chances are good that the first window chosen to view the graph will suggest $y = S(x)$ has the $x$-axis as a horizontal asymptote. From the formula $S(x) = x^2 + \frac{4000}{x}$, however, we get $S(x) \approx x^2$ as $x \to \infty$, so $S(x) \to \infty$. Readjusting the window, we find $S$ does possess a relative minimum at $x \approx 12.60$. As far as we can tell,[4] this is the only relative extremum, and so it is the absolute minimum as well. This means the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.




□

In many instances in the sciences, rational functions are encountered as a result of fundamental natural laws which are typically a result of assuming certain basic relationships between variables. These basic relationships are summarized in the definition below.

Definition 5.4. Suppose $x$, $y$, and $z$ are variable quantities. We say

- $y$ **varies directly with** (or is **directly proportional to**) $x$ if there is a constant $k$ such that $y = kx$.
- $y$ **varies inversely with** (or is **inversely proportional to**) $x$ if there is a constant $k$ such that $y = \frac{k}{x}$.
- $z$ **varies jointly with** (or is **jointly proportional to**) $x$ and $y$ if there is a constant $k$ such that $z = kxy$.

The constant $k$ in the above definitions is called the **constant of proportionality**.

[4] without Calculus, that is...

EXAMPLE 5.3.4. Translate the following into mathematical equations using Definition 5.4.

1. Hooke's Law: The force $F$ exerted on a spring is directly proportional the extension $x$ of the spring.
2. Boyle's Law: At a constant temperature, the pressure $P$ of an ideal gas is inversely proportional to its volume $V$.
3. The volume $V$ of a right circular cone varies jointly with the height $h$ of the cone and the square of the radius $r$ of the base.
4. Ohm's Law: The current $I$ through a conductor between two points is directly proportional to the voltage $V$ between the two points and inversely proportional to the resistance $R$ between the two points.
5. Newton's Law of Universal Gravitation: Suppose two objects, one of mass $m$ and one of mass $M$, are positioned so that the distance between their centers of mass is $r$. The gravitational force $F$ exerted on the two objects varies directly with the product of the two masses and inversely with the square of the distance between their centers of mass.

SOLUTION.

1. Applying the definition of direct variation, we get $F = kx$ for some constant $k$.
2. Since $P$ and $V$ are inversely proportional, we write $P = \frac{k}{V}$.
3. There is a bit of ambiguity here. It's clear the volume and height of the cone is represented by the quantities $V$ and $h$, respectively, but does $r$ represent the radius of the base or the square of the radius of the base? It is the former. Usually, if an algebraic operation is specified (like squaring), it is meant to be expressed in the formula. We apply Definition 5.4 to get $V = khr^2$.
4. Even though the problem doesn't use the phrase 'varies jointly', the fact that the current $I$ is given as relating to two different quantities implies this. Since $I$ varies directly with $V$ but inversely with $R$, we write $I = \frac{kV}{R}$.
5. We write the product of the masses $mM$ and the square of the distance as $r^2$. We have $F$ varies directly with $mM$ and inversely with $r^2$, so that $F = \frac{kmM}{r^2}$. □

### 5.3.1 Exercises

1. Solve each rational equation. Be sure to check for extraneous solutions.

   (a) $\dfrac{x}{5x+4} = 3$

   (b) $\dfrac{3x-1}{x^2+1} = 1$

   (c) $\dfrac{1}{x+3} + \dfrac{1}{x-3} = \dfrac{x^2-3}{x^2-9}$

   (d) $\dfrac{2x+17}{x+1} = x+5$

   (e) $\dfrac{x^2-2x+1}{x^3+x^2-2x} = 1$

   (f) $\dfrac{-x^3+4x}{x^2-9} = 4x$

2. Solve each rational inequality. Express your answer using interval notation.

   (a) $\dfrac{1}{x+2} \geq 0$

   (b) $\dfrac{x-3}{x+2} \leq 0$

   (c) $\dfrac{x}{x^2-1} > 0$

   (d) $\dfrac{4x}{x^2+4} \leq 0$

   (e) $\dfrac{4x}{x^2-4} \geq 0$

   (f) $\dfrac{x^2-x-12}{x^2+x-6} > 0$

   (g) $\dfrac{3x^2-5x-2}{x^2-9} < 0$

   (h) $\dfrac{x^3+2x^2+x}{x^2-x-2} \geq 0$

   (i) $\dfrac{x^2+5x+6}{x^2-1} > 0$

   (j) $\dfrac{3x-1}{x^2+1} \leq 1$

   (k) $\dfrac{2x+17}{x+1} > x+5$

   (l) $\dfrac{-x^3+4x}{x^2-9} \geq 4x$

   (m) $\dfrac{1}{x^2+1} < 0$

   (n) $\dfrac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \geq x$

   (o) $\dfrac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$

3. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[5]

   (a) Find an expression for the volume $V$ of the can based on the height $h$ and the base radius $r$.

   (b) Find an expression for the surface area $S$ of the can based on the height $h$ and the base radius $r$. (Hint: The top and bottom of the can are circles of radius $r$ and the side of the can is really just a rectangle that has been bent into a cylinder.)

   (c) Using the fact that $V = 33.6$, write $S$ as a function of $r$ and state its applied domain.

   (d) Use your graphing calculator to find the dimensions of the can which has minimal surface area.

4. In Exercise 11 in Section 2.2, the population of Sasquatch in Portage County was modeled by the function $P(t) = \frac{150t}{t+15}$, where $t = 0$ represents the year 1803. When were there fewer than 100 Sasquatch in the county?

[5] According to www.dictionary.com, there are different values given for this conversion. We will stick with $33.6\text{in}^3$ for this problem.

5. The cost $C$ in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by $C(p) = \frac{1770p}{100-p}$ where $0 \leq p < 100$.

   (a) Find and interpret $C(25)$ and $C(95)$.

   (b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

   (c) What percentage of the Ippizuti fish can you remove for \$40000?

6. Translate the following into mathematical equations.

   (a) At a constant pressure, the temperature $T$ of an ideal gas is directly proportional to its volume $V$. (This is Charles's Law)

   (b) The frequency of a wave $f$ is inversely proportional to the wavelength of the wave $\lambda$.

   (c) The density $d$ of a material is directly proportional to the mass of the object $m$ and inversely proportional to its volume $V$.

   (d) The square of the orbital period of a planet $P$ is directly proportional to the cube of the semi-major axis of its orbit $a$. (This is Kepler's Third Law of Planetary Motion )

   (e) The drag of an object traveling through a fluid $D$ varies jointly with the density of the fluid $\rho$ and the square of the velocity of the object $\nu$.

   (f) Suppose two electric point charges, one with charge $q$ and one with charge $Q$, are positioned $r$ units apart. The electrostatic force $F$ exerted on the charges varies directly with the product of the two charges and inversely with the square of the distance between the charges. (This is Coulomb's Law)

7. According to this webpage, the frequency $f$ of a vibrating string is given by $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ where $T$ is the tension, $\mu$ is the linear mass[6] of the string and $L$ is the length of the vibrating part of the string. Express this relationship using the language of variation.

8. According to the Centers for Disease Control and Prevention www.cdc.gov, a person's Body Mass Index $B$ is directly proportional to his weight $W$ in pounds and inversely proportional to the square of his height $h$ in inches.

   (a) Express this relationship as a mathematical equation.

   (b) If a person who was 5 feet, 10 inches tall weighed 235 pounds had a Body Mass Index of 33.7, what is the value of the constant of proportionality?

   (c) Rewrite the mathematical equation found in part 8a to include the value of the constant found in part 8b and then find your Body Mass Index.

9. We know that the circumference of a circle varies directly with its radius with $2\pi$ as the constant of proportionality. (That is, we know $C = 2\pi r$.) With the help of your classmates, compile a list of other basic geometric relationships which can be seen as variations.

[6] Also known as the linear density. It is simply a measure of mass per unit length.

### 5.3.2 Answers

1. (a) $x = -\frac{6}{7}$
   (b) $x = 1,\ x = 2$
   (c) $x = -1$
   (d) $x = -6,\ x = 2$
   (e) No solution
   (f) $x = 0,\ x = \pm 2\sqrt{2}$

2. (a) $(-2, \infty)$
   (b) $(-2, 3]$
   (c) $(-1, 0) \cup (1, \infty)$
   (d) $(-\infty, 0]$
   (e) $(-2, 0] \cup (2, \infty)$
   (f) $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
   (g) $\left(-3, -\frac{1}{3}\right) \cup (2, 3)$
   (h) $(-1, 0] \cup (2, \infty)$
   (i) $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$
   (j) $(-\infty, 1] \cup [2, \infty)$
   (k) $(-\infty, -6) \cup (-1, 2)$
   (l) $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$
   (m) No solution
   (n) $[-3, 0) \cup (0, 4) \cup [5, \infty)$
   (o) $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$

3. (a) $V = \pi r^2 h$.
   (b) $S = 2\pi r^2 + 2\pi r h$
   (c) $S(r) = 2\pi r^2 + \frac{67.2}{r}$, Domain $r > 0$
   (d) $r \approx 1.749$ in. and $h \approx 3.498$ in.

4. $P(30) = 100$ so before 1903 there were fewer than 100 Sasquatch in Portage County.

5. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.
   (b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.
   (c) For \$40000 you could remove about 95.76% of the fish.

6. (a) $T = kV$
   (b) [7] $f = \frac{k}{\lambda}$.
   (c) $d = \frac{km}{V}$
   (d) $P^2 = ka^3$
   (e) [8] $D = k\rho\nu^2$.
   (f) [9] $F = \frac{kqQ}{r^2}$

7. Rewriting $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ as $f = \frac{\frac{1}{2}\sqrt{T}}{L\sqrt{\mu}}$ we see that the frequency $f$ varies directly with the square root of the tension and varies inversely with the length and the square root of the linear mass.

8. (a) $B = \frac{kW}{h^2}$
   (b) [10] $k = 702.68$
   (c) $B = \frac{703W}{h^2}$

[7] The character $\lambda$ is the lower case Greek letter 'lambda.'
[8] Note: The characters $\rho$ and $\nu$ are the lower case Greek letters 'rho' and 'nu,' respectively.
[9] Note the similarity to this formula and Newton's Law of Universal Gravitation as discussed in Example 5.
[10] The CDC uses 703.

# Chapter 6

# Further Topics in Functions

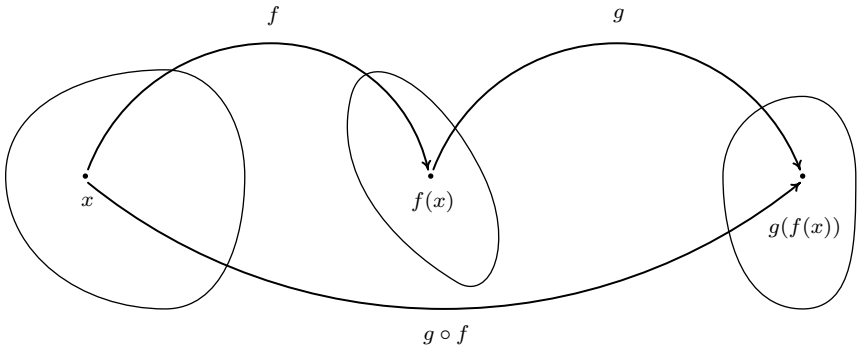
## 6.1 Function Composition

Before we embark upon any further adventures with functions, we need to take some time to gather our thoughts and gain some perspective. Chapter 2 first introduced us to functions in Section 2.1. At that time, functions were specific kinds of relations - sets of points in the plane which passed the Vertical Line Test, Theorem 2.1. In Section 2.2, we developed the idea that functions are processes - rules which match inputs to outputs - and this gave rise to the concepts of domain and range. We spoke about how functions could be combined in Section 2.3 using the four basic arithmetic operations, took a more detailed look at their graphs in Section 2.4 and studied how their graphs behaved under certain classes of transformations in Section 2.5. In Chapter 3, we took a closer look linear functions (Section 3.1), and quadratic functions (Section 3.3). Linear and quadratic functions were special cases of polynomial functions, which we studied in generality in Chapter 4. (In Chapter 10 we will learn the Real Factorization Theorem, Theorem 10.5, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions.) Our next step was to enlarge our field[1] of study to rational functions in Chapter 5. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Chapters 3, 4, and 5 can be thought of as an exhaustive study of linear and quadratic functions and their arithmetic combinations as described in Section 2.3. We now wish to study other algebraic functions, such as $f(x) = \sqrt{x}$ and $g(x) = x^{2/3}$, and the purpose of the first two sections of this chapter is to see how these kinds of functions arise from polynomial and rational functions. To that end, we first study a new way to combine functions as defined below.

Definition 6.1. Suppose $f$ and $g$ are two functions. The **composite** of $g$ with $f$, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided $x$ is an element of the domain of $f$ and $f(x)$ is an element of the domain of $g$.

[1] This is a really bad math pun.

The quantity $g \circ f$ is also read '$g$ composed with $f$' or, more simply '$g$ of $f$.' At its most basic level, Definition 6.1 tells us to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of $x$ in the formula for $g(x)$ with the formula we have for $f(x)$. If we take a step back and look at this from a procedural, 'inputs and outputs' perspective, Defintion 6.1 tells us the output from $g \circ f$ is found by taking the output from $f$, $f(x)$, and then making that the input to $g$. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input $x$ and first applying the procedure $f$ then applying the procedure $g$. Abstractly, we have




In the expression $g(f(x))$, the function $f$ is often called the 'inside' function while $g$ is often called the 'outside' function. There are two ways to go about evaluating composite functions - 'inside out' and 'outside in' - depending on which function we replace with its formula first. Both ways are demonstrated in the following example.

EXAMPLE 6.1.1. Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$. Find and simplify the indicated composite functions. State the domain of each.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $(h \circ h)(x)$
6. $(h \circ (g \circ f))(x)$
7. $((h \circ g) \circ f)(x)$

SOLUTION.

1. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate the two ways to evaluate this.

   - *inside out*: We insert the expression $f(x)$ into $g$ first to get

     $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

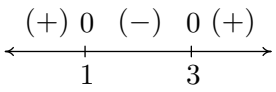
     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

- *outside in*: We use the formula for $g$ first to get

$$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

To find the domain of $g \circ f$, we need to find the elements in the domain of $f$ whose outputs $f(x)$ are in the domain of $g$. We accomplish this by following the rule set forth in Section 2.2, that is, we find the domain **before** we simplify. To that end, we examine $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$. To keep the square root happy, we solve the inequality $x^2 - 4x + 3 \geq 0$ by creating a sign diagram. If we let $r(x) = x^2 - 4x + 3$, we find the zeros of $r$ to be $x = 1$ and $x = 3$. We obtain




Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

2. To find $(f \circ g)(x)$, we find $f(g(x))$.

- *inside out*: We insert the expression $g(x)$ into $f$ first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(2 - \sqrt{x+3}\right) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= 4 - 4\sqrt{x+3} + \left(\sqrt{x+3}\right)^2 - 8 + 4\sqrt{x+3} \\
&= 4 + x + 3 - 8 \\
&= x - 1
\end{aligned}$$

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= (g(x))^2 - 4\,(g(x)) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= x - 1 \qquad \text{same algebra as before}
\end{aligned}$$

Thus we get $(f \circ g)(x) = x - 1$. To find the domain of $(f \circ g)$, we look to the step before

we did any simplification and find $(f \circ g)(x) = \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right)$. To keep the square root happy, we set $x + 3 \geq 0$ and find our domain to be $[-3, \infty)$.

3. To find $(g \circ h)(x)$, we compute $g(h(x))$.

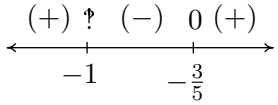
   - *inside out*: We insert the expression $h(x)$ into $g$ first to get

$$\begin{aligned}
(g \circ h)(x) &= g(h(x)) \\
&= g\left(\frac{2x}{x+1}\right) \\
&= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\
&= 2 - \sqrt{\frac{2x}{x+1} + \frac{3(x+1)}{x+1}} \quad \text{get common denominators} \\
&= 2 - \sqrt{\frac{5x+3}{x+1}}
\end{aligned}$$

   - *outside in*: We use the formula for $g(x)$ first to get

$$\begin{aligned}
(g \circ h)(x) &= g(h(x)) \\
&= 2 - \sqrt{h(x) + 3} \\
&= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\
&= 2 - \sqrt{\frac{5x+3}{x+1}} \qquad \text{get common denominators as before}
\end{aligned}$$

   Hence, $(g \circ h)(x) = 2 - \sqrt{\frac{5x+3}{x+1}}$. To find the domain, we look to the step before we began to simplify: $(g \circ h)(x) = 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3}$. To avoid division by zero, we need $x \neq -1$. To keep the radical happy, we need to solve $\frac{2x}{x+1} + 3 \geq 0$. Getting common denominators as before, this reduces to $\frac{5x+3}{x+1} \geq 0$. Defining $r(x) = \frac{5x+3}{x+1}$, we have that $r$ is undefined at $x = -1$ and $r(x) = 0$ at $x = -\frac{3}{5}$. We get




   Our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

4. We find $(h \circ g)(x)$ by finding $h(g(x))$.

   - *inside out*: We insert the expression $g(x)$ into $h$ first to get

$$\begin{aligned}(h \circ g)(x) &= h(g(x)) \\ &= h\left(2 - \sqrt{x+3}\right) \\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}\end{aligned}$$

   - *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned}(h \circ g)(x) &= h(g(x)) \\ &= \frac{2\left(g(x)\right)}{\left(g(x)\right) + 1} \\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}\end{aligned}$$

   Hence, $(h \circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. To find the domain of $h \circ g$, we look to the step before any simplification: $(h \circ g)(x) = \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}$. To keep the square root happy, we require $x + 3 \geq 0$ or $x \geq -3$. Setting the denominator equal to zero gives $\left(2 - \sqrt{x+3}\right) + 1 = 0$ or $\sqrt{x+3} = 3$. Squaring both sides gives us $x + 3 = 9$, or $x = 6$. Since $x = 6$ checks in the original equation, $\left(2 - \sqrt{x+3}\right) + 1 = 0$, we know $x = 6$ is the only zero of the denominator. Hence, the domain of $h \circ g$ is $[-3, 6) \cup (6, \infty)$.

5. To find $(h \circ h)(x)$, we substitute the function $h$ into itself, $h(h(x))$.

   - *inside out*: We insert the expression $h(x)$ into $h$ to get

$$\begin{aligned}(h \circ h)(x) &= h(h(x)) \\ &= h\left(\frac{2x}{x+1}\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{2\left(\dfrac{2x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right)+1} \\
&= \frac{\dfrac{4x}{x+1}}{\dfrac{2x}{x+1}+1} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{\dfrac{4x}{x+1} \cdot (x+1)}{\left(\dfrac{2x}{x+1}\right) \cdot (x+1) + 1 \cdot (x+1)} \\
&= \frac{\dfrac{4x}{\cancel{(x+1)}} \cdot \cancel{(x+1)}}{\dfrac{2x}{\cancel{(x+1)}} \cdot \cancel{(x+1)} + x + 1} \\
&= \frac{4x}{3x+1}
\end{aligned}$$

- *outside in*: This approach yields

$$\begin{aligned}
(h \circ h)(x) &= h(h(x)) \\
&= \frac{2(h(x))}{h(x)+1} \\
&= \frac{2\left(\dfrac{2x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right)+1} \\
&= \frac{4x}{3x+1} \qquad \text{same algebra as before}
\end{aligned}$$

  To find the domain of $h \circ h$, we analyze $(h \circ h)(x) = \dfrac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1}$. To keep the denominator $x+1$ happy, we need $x \neq -1$. Setting the denominator $\frac{2x}{x+1}+1 = 0$ gives $x = -\frac{1}{3}$. Our domain is $(-\infty,-1) \cup \left(-1,-\frac{1}{3}\right) \cup \left(-\frac{1}{3},\infty\right)$.

6. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function $h$ with the result. We know from number 1 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$. We now proceed as usual.

- *inside out*: We insert the expression $(g \circ f)(x)$ into $h$ first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

So we get $(h \circ (g \circ f))(x) = \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}$. To find the domain, we look at the step before we began to simplify, $(h \circ (g \circ f))(x) = \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1}$. For the square root, we need $x^2 - 4x + 3 \geq 0$, which we determined in number 1 to be $(-\infty, 1] \cup [3, \infty)$. Next, we set the denominator to zero and solve: $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. To solve $x^2 - 4x - 6 = 0$, we use the quadratic formula and get $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$. Hence we must exclude these numbers from the domain of $h \circ (g \circ f)$. Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

7. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with $f$. From number 4, we gave $(h \circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. We now proceed as before.

- *inside out*: We insert the expression $f(x)$ into $h \circ g$ first to get

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x))$$

$$\begin{aligned} &= (h \circ g)\left(x^2 - 4x\right) \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(x)$ first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}} \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

We note that the formula for $((h \circ g) \circ f)(x)$ before simplification is identical to that of $(h \circ (g \circ f))(x)$ before we simplified it. Hence, the two functions have the same domain, $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

□

It should be clear from Example 6.1.1 that, in general, when you compose two functions, such as $f$ and $g$ above, the order matters.[2] We found that the functions $f \circ g$ and $g \circ f$ were different as were $g \circ h$ and $h \circ g$. Thinking of functions as processes, this isn't all that surprising. If we think of one process as putting on our socks, and the other as putting on our shoes, the order in which we do these two tasks does matter.[3] Also note the importance of finding the domain of the composite function **before** simplifying. For instance, the domain of $f \circ g$ is much different than its simplified formula would indicate. Composing a function with itself, as in the case of $h \circ h$, may seem odd. Looking at this from a procedural perspective, however, this merely indicates performing a task $h$ and then doing it again - like setting the washing machine to do a 'double rinse'. Composing a function with itself is called 'iterating' the function, and we could easily spend an entire course on just that. The last two problems in Example 6.1.1 serve to demonstrate the **associative** property of functions. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. This property as well as another important property are listed in the theorem below.

[2] This shows us function composition isn't **commutative**. An example of an operation we perform on two functions which is commutative is function addition, which we defined in Section 2.3. In other words, the functions $f + g$ and $g + f$ are always equal. Which of the remaining operations on functions we have discussed are commutative?

[3] A more mathematical example in which the order of two processes matters can be found in Section 2.5. In fact, all of the transformations in that section can be viewed in terms of composing functions with linear functions.

Theorem 6.1. **Properties of Function Composition:** Suppose $f$, $g$, and $h$ are functions.

- $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- If $I$ is defined as $I(x) = x$ for all real numbers $x$, then $I \circ f = f \circ I = f$.

By repeated applications of Definition 6.1, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 6.1. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise 12 in Section 3.1. If we compose the function $I$ with a function $f$, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(x)$. This establishes that we have an identity for function composition much in the same way the real number 1 is an identity for real number multiplication. That is, just as for any real number $x$, $1 \cdot x = x \cdot 1 = x$ , we have for any function $f$, $I \circ f = f \circ I = f$. We shall see the concept of an identity take on great significance in the next section. Out in the wild, function composition is often used to relate two quantities which may not be directly related, but have a variable in common, as illustrated in our next example.

Example 6.1.2. The surface area $S$ of a sphere is a function of its radius $r$ and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where $t$ is measured in seconds, $t \geq 0$, and $r$ is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, $t$, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area $S$ is ultimately a function of time $t$ and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'middle man' $r$. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

Example 6.1.3. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x) = |3x - 1|$
2. $G(x) = \dfrac{2}{x^2 + 1}$

3. $H(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$

SOLUTION. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function $F$ as $F = g \circ f$ for functions $g$ and $f$. From Definition 6.1, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and $g$ as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$, so we are done.

2. We attack deconstructing $G$ from an operational approach. Given an input $x$, the first step is to square $x$, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write $G$ as a composite of three functions: $f$, $g$ and $h$. Our first function, $f$, is the function that squares its input, $f(x) = x^2$. The next function is the function that adds 1 to its input, $g(x) = x + 1$. Our last function takes its input and divides it into 2, $h(x) = \frac{2}{x}$. The claim is that $G = h \circ g \circ f$. We find $(h \circ g \circ f)(x) = h(g(f(x))) = h(g\left(x^2\right)) = h\left(x^2 + 1\right) = \frac{2}{x^2+1} = G(x)$.

3. If we look $H(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{x}$ is a simple piece of the larger function. If we define $f(x) = \sqrt{x}$, we have $H(x) = \frac{f(x)+1}{f(x)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula from $g(x)$ by looking at what is being done to $f(x)$. We find $g(x) = \frac{x+1}{x-1}$. We check $(g \circ f)(x) = g(f(x)) = \frac{f(x)+1}{f(x)-1} = \frac{\sqrt{x}+1}{\sqrt{x}-1} = H(x)$, as required. □

### 6.1.1 Exercises

1. Let $f(x) = 3x - 6$, $g(x) = |x|$, $h(x) = \sqrt{x}$ and $k(x) = \dfrac{1}{x}$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ k)(x)$
   (h) $(k \circ f)(x)$
   (i) $(h \circ k)(x)$
   (j) $(k \circ h)(x)$
   (k) $(f \circ g \circ h)(x)$
   (l) $(h \circ g \circ k)(x)$
   (m) $(k \circ h \circ f)(x)$
   (n) $(h \circ k \circ g \circ f)(x)$

2. Let $f(x) = 2x + 1$, $g(x) = x^2 - x - 6$ and $h(x) = \dfrac{x+6}{x-6}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(g \circ f)(x)$
   (b) $(h \circ f)(x)$
   (c) $(h \circ g)(x)$
   (d) $(h \circ h)(x)$

3. Let $f(x) = \sqrt{x-3}$, $g(x) = 4x + 3$ and $h(x) = \dfrac{x-2}{x+3}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ f)(x)$
   (h) $(g \circ g)(x)$
   (i) $(h \circ h)(x)$

4. Let $f(x) = \sqrt{9 - x^2}$ and $g(x) = x^2 - 9$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ f)(x)$
   (b) $(g \circ g)(x)$
   (c) $(g \circ f)(x)$
   (d) $(f \circ g)(x)$

5. Let $f$ be the function defined by $f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$ and let $g$ be the function defined $g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$.

   Find the each of the following values if it exists.

   (a) $(f \circ g)(3)$
   (b) $f(g(-1))$
   (c) $(f \circ f)(0)$
   (d) $(f \circ g)(-3)$
   (e) $(g \circ f)(3)$
   (f) $g(f(-3))$
   (g) $(g \circ g)(-2)$
   (h) $(g \circ f)(-2)$
   (i) $g(f(g(0)))$
   (j) $f(f(f(-1)))$
   (k) $f(f(f(f(f(1)))))$
   (l) $\overbrace{(g \circ g \circ \cdots \circ g)}^{n \text{ times}}(0)$

6. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

7. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

8. Write the following as a composition of two or more non-identity functions.

   (a) $h(x) = \sqrt{2x-1}$
   (b) $r(x) = \dfrac{2}{5x+1}$
   (c) $F(x) = \left(x^2-1\right)^3$
   (d) $R(x) = \dfrac{2x^3+1}{x^3-1}$

9. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

10. The volume $V$ of a cube is a function of its side length $x$. Let's assume that $x = t+1$ is also a function of time $t$, where $x$ is measured in inches and $t$ is measured in minutes. Find a formula for $V$ as a function of $t$.

11. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour $x$ is given by $x(t) = -4t^2 + 20t + 92$, where $t$ is the number of hours since 10 AM, $0 \le t \le 4$.

    (a) Find an expression for the revenue per hour $R$ as a function of $x$.
    (b) Find and simplify $(R \circ x)(t)$. What does this represent?
    (c) What is the revenue per hour at noon?

12. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

### 6.1.2 Answers

1. (a) $(f \circ g)(x) = 3|x| - 6$
   Domain: $(-\infty, \infty)$
   (b) $(g \circ f)(x) = |3x - 6|$
   Domain: $(-\infty, \infty)$
   (c) $(f \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$
   (d) $(h \circ f)(x) = \sqrt{3x - 6}$
   Domain: $[2, \infty)$
   (e) $(g \circ h)(x) = \sqrt{x}$
   Domain: $[0, \infty)$
   (f) $(h \circ g)(x) = \sqrt{|x|}$
   Domain: $(-\infty, \infty)$
   (g) $(f \circ k)(x) = \dfrac{3}{x} - 6$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   (h) $(k \circ f)(x) = \dfrac{1}{3x - 6}$
   Domain: $(-\infty, 2) \cup (2, \infty)$
   (i) $(h \circ k)(x) = \sqrt{\dfrac{1}{x}}$
   Domain: $(0, \infty)$
   (j) $(k \circ h)(x) = \dfrac{1}{\sqrt{x}}$
   Domain: $(0, \infty)$
   (k) $(f \circ g \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$
   (l) $(h \circ g \circ k)(x) = \sqrt{\left|\dfrac{1}{x}\right|}$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   (m) $(k \circ h \circ f)(x) = \dfrac{1}{\sqrt{3x - 6}}$
   Domain: $(2, \infty)$
   (n) $(h \circ k \circ g \circ f)(x) = \sqrt{\dfrac{1}{|3x - 6|}}$
   Domain: $(-\infty, 2) \cup (2, \infty)$

2. (a) $(g \circ f)(x) = 4x^2 + 2x - 6$
   Domain: $(-\infty, \infty)$
   (b) $(h \circ f)(x) = \dfrac{2x + 7}{2x - 5}$
   Domain: $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$
   (c) $(h \circ g)(x) = \dfrac{x^2 - x}{x^2 - x - 12}$
   Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$
   (d) $(h \circ h)(x) = -\dfrac{7x - 30}{5x - 42}$
   Domain: $(-\infty, 6) \cup \left(6, \frac{42}{5}\right) \cup \left(\frac{42}{5}, \infty\right)$

3. (a) $(f \circ g)(x) = 2\sqrt{x}$
   Domain: $[0, \infty)$
   (b) $(g \circ f)(x) = 4\sqrt{x - 3} + 3$
   Domain: $[3, \infty)$
   (c) $(f \circ h)(x) = \sqrt{\dfrac{-2x - 11}{x + 3}}$
   Domain: $\left[-\frac{11}{2}, -3\right)$
   (d) $(h \circ f)(x) = \dfrac{\sqrt{x - 3} - 2}{\sqrt{x - 3} + 3}$
   Domain: $[3, \infty)$
   (e) $(g \circ h)(x) = \dfrac{7x + 1}{x + 3}$
   Domain: $(-\infty, -3) \cup (-3, \infty)$
   (f) $(h \circ g)(x) = \dfrac{4x + 1}{4x + 6}$
   Domain: $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$
   (g) $(f \circ f)(x) = \sqrt{\sqrt{x - 3} - 3}$
   Domain: $[12, \infty)$
   (h) $(g \circ g)(x) = 16x + 15$
   Domain: $(-\infty, \infty)$
   (i) $(h \circ h)(x) = \dfrac{-x - 8}{4x + 7}$
   Domain:
   $(-\infty, -3) \cup \left(-3, -\frac{7}{4}\right) \cup \left(-\frac{7}{4}, \infty\right)$

4. (a) $(f \circ f)(x) = |x|$
   Domain: $[-3, 3]$
   (b) $(g \circ g)(x) = x^4 - 18x^2 + 72$
   Domain: $(-\infty, \infty)$
   (c) $(g \circ f)(x) = -x^2$
   Domain: $[-3, 3]$
   (d) $(f \circ g)(x) = \sqrt{-x^4 + 18x^2 - 72}$
   Domain: $[-\sqrt{12}, -\sqrt{6}] \cup [\sqrt{6}, \sqrt{12}]$[4]

5. (a) $(f \circ g)(3) = f(g(3)) = f(2) = 4$
   (b) $f(g(-1)) = f(-4)$ which is undefined
   (c) $(f \circ f)(0) = f(f(0)) = f(1) = 3$
   (d) $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$
   (e) $(g \circ f)(3) = g(f(3)) = g(-1) = -4$
   (f) $g(f(-3)) = g(4)$ which is undefined
   (g) $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$
   (h) $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$
   (i) $g(f(g(0))) = g(f(0)) = g(1) = -3$
   (j) $f(f(f(-1))) = f(f(0)) = f(1) = 3$
   (k) $f(f(f(f(f(1))))) = f(f(f(3))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$
   (l) $\overbrace{(g \circ g \circ \cdots \circ g)}^{n \text{ times}}(0) = 0$

6. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

7. One possible solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

8. (a) $h(x) = (g \circ f)(x)$ where $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$.
   (b) $r(x) = (g \circ f)(x)$ where $f(x) = 5x + 1$ and $g(x) = \dfrac{2}{x}$.
   (c) $F(x) = (g \circ f)(x)$ where $f(x) = x^2 - 1$ and $g(x) = x^3$.
   (d) $R(x) = (g \circ f)(x)$ where $f(x) = x^3$ and $g(x) = \dfrac{2x+1}{x-1}$.

9. $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \dfrac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

10. $V(x) = x^3$ so $V(x(t)) = (t+1)^3$

11. (a) $R(x) = 2x$
    (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \le t \le 4$. This gives the revenue per hour as a function of time.
    (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

[4]The quantity $-x^4 + 18x^2 - 72$ is a 'quadratic in disguise' which factors nicely.

## 6.2 Inverse Functions

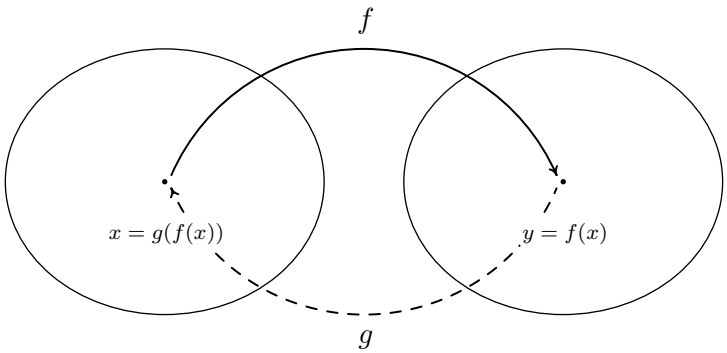
Thinking of a function as a process like we did in Section 2.2, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x) = 3x + 4$. Thinking of $f$ as a process, we start with an input $x$ and apply two steps, as we saw in Section 2.2

1. multiply by 3
2. add 4

To reverse this process, we seek a function $g$ which will undo each of these steps and take the output from $f$, $3x + 4$, and return the input $x$. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function $g$ should undo the second step of $f$ first. That is, the function $g$ should

1. *subtract* 4
2. *divide* by 3

Following this procedure, we get $g(x) = \frac{x-4}{3}$. Let's check to see if the function $g$ does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from $f$, we substitute it into $g$ to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to $f$. To check that $g$ does the job for all $x$ in the domain of $f$, we take the generic output from $f$, $f(x) = 3x + 4$, and substitute that into $g$. That is, $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to $f$. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see $g$ first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does $g$ undo $f$, but $f$ also undoes $g$. That is, if we take the output from $g$, $g(x) = \frac{x-4}{3}$, and put that into $f$, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 6.1, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively. Abstractly, we can visualize the relationship between $f$ and $g$ in the diagram below.



The main idea to get from the diagram is that $g$ takes the outputs from $f$ and returns them to their respective inputs, and conversely, $f$ takes outputs from $g$ and returns them to their respective inputs. We now have enough background to state the central definition of the section.

DEFINITION 6.2. Suppose $f$ and $g$ are two functions such that

1. $(g \circ f)(x) = x$ for all $x$ in the domain of $f$ **and**
2. $(f \circ g)(x) = x$ for all $x$ in the domain of $g$.

Then $f$ and $g$ are said to be **inverses** of each other. The functions $f$ and $g$ are said to be **invertible**.

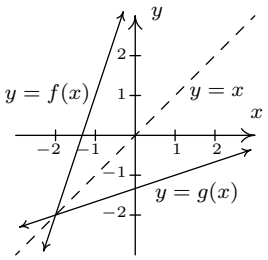
Our first result of the section formalizes the concepts that inverse functions exchange inputs and outputs and is a consequence of Definition 6.2 and the Fundamental Graphing Principle for Functions.

THEOREM 6.2. **Properties of Inverse Functions:** Suppose $f$ and $g$ are inverse functions.

- The range[a] of $f$ is the domain of $g$ and the domain of $f$ is the range of $g$
- $f(a) = b$ if and only if $g(b) = a$
- $(a, b)$ is on the graph of $f$ if and only if $(b, a)$ is on the graph of $g$

[a] Recall this is the set of all outputs of a function.

The third property in Theorem 6.2 tells us that the graphs of inverse functions are reflections about the line $y = x$. For a proof of this, we refer the reader to Example 1.1.6 in Section 1.1. A plot of the inverse functions $f(x) = 3x + 4$ and $g(x) = \frac{x-4}{3}$ confirms this to be the case.




If we abstract one step further, we can express the sentiment in Definition 6.2 by saying that $f$ and $g$ are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where $I_1$ is the identity function

restricted[1] to the domain of $f$ and $I_2$ is the identity function restricted to the domain of $g$. In other words, $I_1(x) = x$ for all $x$ in the domain of $f$ and $I_2(x) = x$ for all $x$ in the domain of $g$. Using this description of inverses along with the properties of function composition listed in Theorem 6.1, we can show that function inverses are unique.[2] Suppose $g$ and $h$ are both inverses of a function $f$. By Theorem 6.2, the domain of $g$ is equal to the domain of $h$, since both are the range of $f$. This means the identity function $I_2$ applies both to the domain of $h$ and the domain of $g$. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.[3] We summarize the discussion of the last two paragraphs in the following theorem.[4]

Theorem 6.3. **Uniqueness of Inverse Functions and Their Graphs :** Suppose $f$ is an invertible function.

- There is exactly one inverse function for $f$, denoted $f^{-1}$ (read $f$-inverse)
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

The notation $f^{-1}$ is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely **not** the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$. Why does this confusing notation persist? As we mentioned in Section 6.1, the identity function $I$ is to function composition what the real number 1 is to real number multiplication. The choice of notation $f^{-1}$ alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

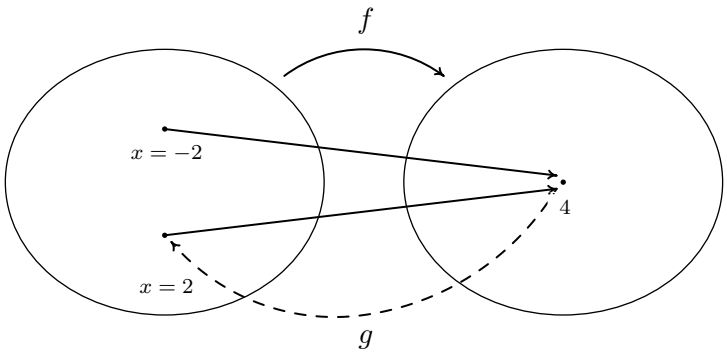
Let's turn our attention to the function $f(x) = x^2$. Is $f$ invertible? A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. Checking the composition yields $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to $x$ for all $x$ in the domain $(-\infty, \infty)$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$, which means $g$ failed to return the input $-2$ from its output 4. What $g$ did, however, is match the output 4 to a **different** input, namely 2, which satisfies $f(2) = 4$. This issue is presented schematically in the picture below.

[1] The identity function $I$, which was introduced in Section 3.1 and mentioned in Theorem 6.1, has a domain of all real numbers. In general, the domains of $f$ and $g$ are not all real numbers, which necessitates the restrictions listed here.

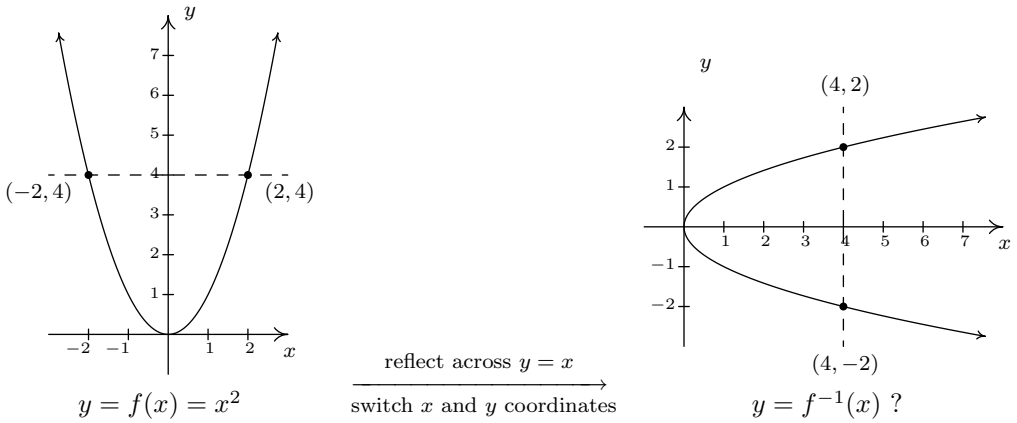
[2] In other words, invertible functions have exactly one inverse.

[3] It is an excellent exercise to explain each step in this string of equalities.

[4] In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.




We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a **function** which takes 4 back to **both** $x = 2$ and $x = -2$. (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if $y = f^{-1}(x)$ exists, its graph can be obtained by reflecting $y = x^2$ about the line $y = x$, in accordance with Theorem 6.3. Doing so produces




We see that the line $x = 4$ intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, Theorem 2.1, and as such, does not represent $y$ as a function of $x$. The vertical line $x = 4$ on the graph on the right corresponds to the *horizontal line* $y = 4$ on the graph of $y = f(x)$. The fact that the horizontal line $y = 4$ intersects the graph of $f$ twice means two **different** inputs, namely $x = -2$ and $x = 2$, are matched with the **same** output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, **different** inputs must go to **different** outputs, or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

DEFINITION 6.3. A function $f$ is said to be **one-to-one** if $f$ matches different inputs to different outputs. Equivalently, $f$ is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

Theorem 6.4. **The Horizontal Line Test:** A function $f$ is one-to-one if and only if no horizontal line intersects the graph of $f$ more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if $f$ is invertible, then $f$ must be one-to-one, otherwise the graph given by reflecting the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of $f$ as the set of ordered pairs which constitute its graph. If switching the $x$- and $y$-coordinates of the points results in a function, then $f$ is invertible and we have found $f^{-1}$. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes $x$ as a function of $y$. We summarize these results below.

Theorem 6.5. **Equivalent Conditions for Invertibility:** Suppose $f$ is a function. The following statements are equivalent.

- $f$ is invertible.
- $f$ is one-to-one.
- The graph of $f$ passes the Horizontal Line Test.

We put this result to work in the next example.

Example 6.2.1. Determine if the following functions are one-to-one in two ways: (a) analytically using Definition 6.3 and (b) graphically using the Horizontal Line Test.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$
3. $h(x) = x^2 - 2x + 4$
4. $F = \{(-1, 1), (0, 2), (2, 1)\}$

Solution.

1. (a) To determine if $f$ is one-to-one analytically, we assume $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, $f$ is one-to-one.

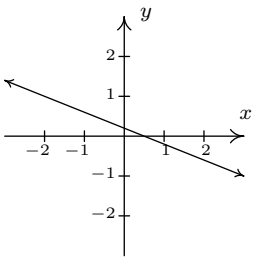
(b) To check if $f$ is one-to-one graphically, we look to see if the graph of $y = f(x)$ passes the Horizontal Line Test. We have that $f$ is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of $f$ passes the Horizontal Line Test as seen below.

2. (a) We begin with the assumption that $g(c) = g(d)$ and try to show $c = d$.

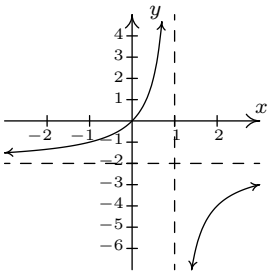
$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c-2cd &= 2d-2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that $g$ is one-to-one.

(b) We can graph $g$ using the six step procedure outlined in Section 5.2. We get the sole intercept at $(0, 0)$, a vertical asymptote $x = 1$ and a horizontal asymptote (which the graph never crosses) $y = -2$. We see from that the graph of $g$ passes the Horizontal Line Test.




$y = f(x)$




$y = g(x)$

3. (a) We begin with $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

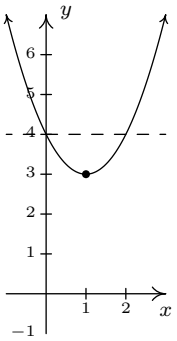
$$\begin{aligned} h(c) &= h(d) \\ c^2 - 2c + 4 &= d^2 - 2d + 4 \\ c^2 - 2c &= d^2 - 2d \\ c^2 - d^2 - 2c + 2d &= 0 \\ (c+d)(c-d) - 2(c-d) &= 0 \\ (c-d)((c+d) - 2) &= 0 \qquad \text{factor by grouping} \\ c - d = 0 \quad &\text{or} \quad c + d - 2 = 0 \\ c = d \quad &\text{or} \quad c = 2 - d \end{aligned}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that $f$ may not be one-to-one. Taking $d = 0$, we get $c = 0$ or $c = 2$. With $f(0) = 4$ and $f(2) = 4$, we have produced two different inputs with the same output meaning $f$ is not one-to-one.

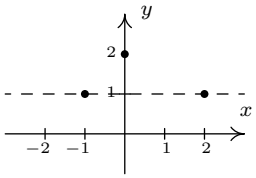
(b) We note that $h$ is a quadratic function and we graph $y = h(x)$ using the techniques presented in Section 3.3. The vertex is $(1, 3)$ and the parabola opens upwards. We see immediately from the graph that $h$ is not one-to-one, since there are several horizontal lines which cross the graph more than once.

4. (a) The function $F$ is given to us as a set of ordered pairs. The condition $F(c) = F(d)$ means the outputs from the function (the $y$-coordinates of the ordered pairs) are the same. We see that the points $(-1, 1)$ and $(2, 1)$ are both elements of $F$ with $F(-1) = 1$ and $F(2) = 1$. Since $-1 \neq 2$, we have established that $F$ is **not** one-to-one.

(b) Graphically, we see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of $F$ fails the Horizontal Line Test.



$y = h(x)$



$y = F(x)$

□

We have shown that the functions $f$ and $g$ in Example 6.2.1 are one-to-one. This means they are invertible, so it is natural to wonder what $f^{-1}(x)$ and $g^{-1}(x)$ would be. For $f(x) = \frac{1-2x}{5}$, we can think our way through the inverse since there is only one occurrence of $x$. We can track step-by-step what is done to $x$ and reverse those steps as we did at the beginning of the chapter. The function $g(x) = \frac{2x}{1-x}$ is a bit trickier since $x$ occurs in two places. When one evaluates $g(x)$ for a specific value of $x$, which is first, the $2x$ or the $1-x$? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem 6.2 tells us equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

**Steps for finding the Inverse of a One-to-one Function**

1. Write $y = f(x)$
2. Interchange $x$ and $y$
3. Solve $x = f(y)$ for $y$ to obtain $y = f^{-1}(x)$

Note that we could have simply written 'Solve $x = f(y)$ for $y$' and be done with it. The act of interchanging the $x$ and $y$ is there to remind us that we are finding the inverse function by switching the inputs and outputs.

EXAMPLE 6.2.2. Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$

SOLUTION.

1. As we mentioned earlier, it is possible to think our way through the inverse of $f$ by recording the steps we apply to $x$ and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write $y = f(x)$ and proceed to switch $x$ and $y$

$$\begin{aligned} y &= f(x) \\ y &= \frac{1-2x}{5} \\ x &= \frac{1-2y}{5} \qquad \text{switch } x \text{ and } y \\ 5x &= 1-2y \\ 5x-1 &= -2y \\ \frac{5x-1}{-2} &= y \\ y &= -\tfrac{5}{2}x+\tfrac{1}{2} \end{aligned}$$

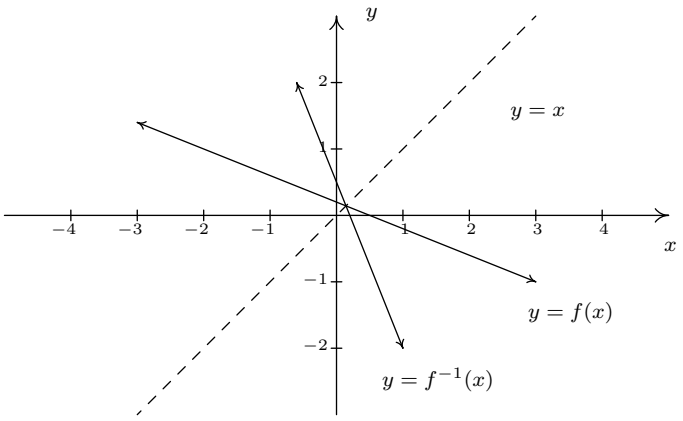
We have $f^{-1}(x) = -\frac{5}{2}x+\frac{1}{2}$. To check this answer analytically, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all $x$ in the domain of $f$, which is all real numbers.

$$\begin{aligned} \left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= -\tfrac{5}{2}f(x)+\tfrac{1}{2} \\ &= -\tfrac{5}{2}\left(\frac{1-2x}{5}\right)+\tfrac{1}{2} \\ &= -\tfrac{1}{2}(1-2x)+\tfrac{1}{2} \\ &= -\tfrac{1}{2}+x+\tfrac{1}{2} \\ &= x \checkmark \end{aligned}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the range of $f$ which is also all real numbers. (Recall that the domain of $f^{-1}$) is the range of $f$.)

$$\begin{aligned} \left(f \circ f^{-1}\right)(x) &= f(f^{-1}(x)) \\ &= \frac{1-2f^{-1}(x)}{5} \\ &= \frac{1-2\left(-\frac{5}{2}x+\frac{1}{2}\right)}{5} \\ &= \frac{1+5x-1}{5} \\ &= \frac{5x}{5} \\ &= x \checkmark \end{aligned}$$

To check our answer graphically, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.[5] They appear to be reflections across the line $y = x$.




2. To find $g^{-1}(x)$, we start with $y = g(x)$. We note that the domain of $g$ is $(-\infty, 1) \cup (1, \infty)$.

$$\begin{aligned} y &= g(x) \\ y &= \frac{2x}{1-x} \\ x &= \frac{2y}{1-y} \qquad \text{switch } x \text{ and } y \\ x(1-y) &= 2y \\ x - xy &= 2y \\ x &= xy + 2y \\ x &= y(x+2) \qquad \text{factor} \\ y &= \frac{x}{x+2} \end{aligned}$$

We obtain $g^{-1}(x) = \frac{x}{x+2}$. To check this analytically, we first check $\left(g^{-1} \circ g\right)(x) = x$ for all $x$ in the domain of $g$, that is, for all $x \neq 1$.

$$\begin{aligned} \left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) \\ &= g^{-1}\left(\frac{2x}{1-x}\right) \end{aligned}$$

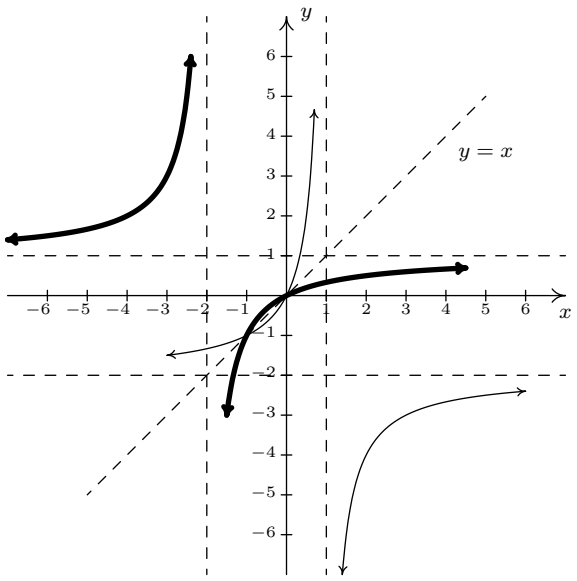
[5] Note that if you perform your check on a calculator for more sophisticated functions, you'll need to take advantage of the 'ZoomSquare' feature to get the correct geometric perspective.

$$
\begin{aligned}
&= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \\
&= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} \quad \text{clear denominators} \\
&= \frac{2x}{2x+2(1-x)} \\
&= \frac{2x}{2x+2-2x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}
$$

Next, we check $g\left(g^{-1}(x)\right) = x$ for all $x$ in the range of $g$. From the graph of $g$ in Example 6.2.1, we have that the range of $g$ is $(-\infty, -2) \cup (-2, \infty)$. This matches the domain we get from the formula $g^{-1}(x) = \frac{x}{x+2}$, as it should.

$$\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} \quad \text{clear denominators} \\
&= \frac{2x}{(x+2)-x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}$$

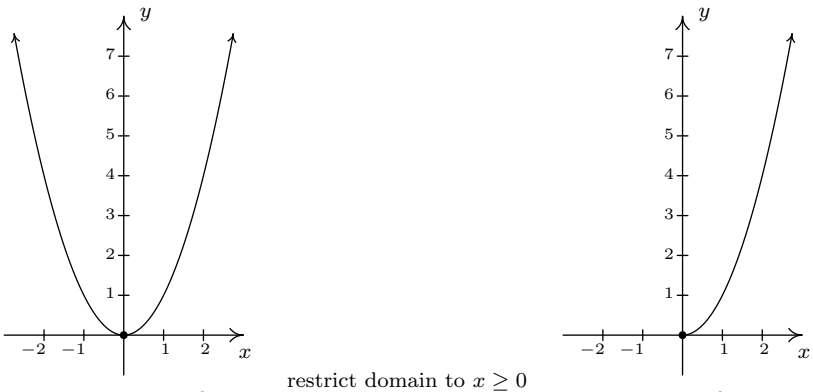
Graphing $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for $y = g^{-1}(x)$. Note that the vertical asymptote $x = 1$ of the graph of $g$ corresponds to the horizontal asymptote $y = 1$ of the graph of $g^{-1}$, as it should since $x$ and $y$ are switched. Similarly, the horizontal asymptote $y = -2$ of the graph of $g$ corresponds to the vertical asymptote $x = -2$ of the graph of $g^{-1}$.




$y = g(x)$ and $\boldsymbol{y = g^{-1}(x)}$

□

We now return to $f(x) = x^2$. We know that $f$ is not one-to-one, and thus, is not invertible. However, if we restrict the domain of $f$, we can produce a new function $g$ which is one-to-one. If we define $g(x) = x^2$, $x \geq 0$, then we have

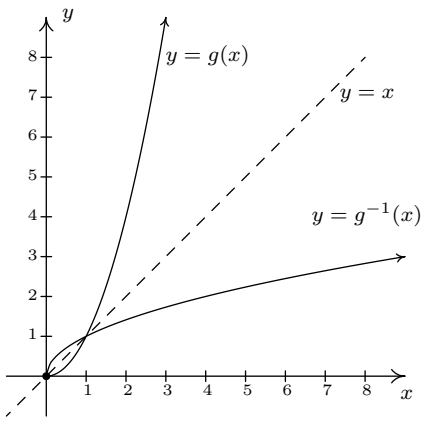



$y = f(x) = x^2$ $y = g(x) = x^2$, $x \geq 0$

The graph of $g$ passes the Horizontal Line Test. To find an inverse of $g$, we proceed as usual

$$\begin{aligned} y &= g(x) \\ y &= x^2, \ x \geq 0 \\ x &= y^2, \ y \geq 0 \quad \text{switch } x \text{ and } y \\ y &= \pm\sqrt{x} \\ y &= \sqrt{x} \qquad\qquad \text{since } y \geq 0 \end{aligned}$$

We get $g^{-1}(x) = \sqrt{x}$. At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on $g$ saves the day. We get $\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) = g^{-1}\left(x^2\right) = \sqrt{x^2} = |x| = x$, since $x \geq 0$. Checking $\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$. Graphing[6] $g$ and $g^{-1}$ on the same set of axes shows that they are reflections about the line $y = x$.




Our next example continues the theme of domain restriction.

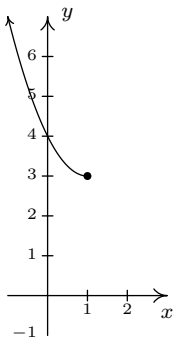
EXAMPLE 6.2.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(x) = \sqrt{x+2} - 1$

SOLUTION.

1. The function $j$ is a restriction of the function $h$ from Example 6.2.1. Since the domain of $j$ is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. We see that the graph of $j$ passes the Horizontal Line Test and thus $j$ is invertible.

[6] We graphed $y = \sqrt{x}$ in Section 2.5.




$y = j(x)$

We now use our algorithm to find $j^{-1}(x)$.

$$
\begin{aligned}
y &= j(x) \\
y &= x^2 - 2x + 4, \;\; x \leq 1 \\
x &= y^2 - 2y + 4, \;\; y \leq 1 && \text{switch } x \text{ and } y \\
0 &= y^2 - 2y + 4 - x \\
y &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} && \text{quadratic formula, } c = 4 - x \\
y &= \frac{2 \pm \sqrt{4x - 12}}{2} \\
y &= \frac{2 \pm \sqrt{4(x-3)}}{2} \\
y &= \frac{2 \pm 2\sqrt{x-3}}{2} \\
y &= \frac{2\left(1 \pm \sqrt{x-3}\right)}{2} \\
y &= 1 \pm \sqrt{x-3} \\
y &= 1 - \sqrt{x-3} && \text{since } y \leq 1.
\end{aligned}
$$

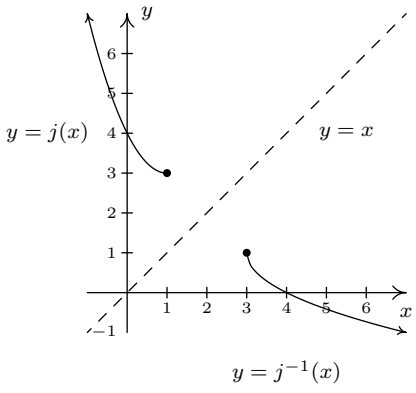
We have $j^{-1}(x) = 1 - \sqrt{x-3}$. When we simplify $\left(j^{-1} \circ j\right)(x)$, we need to remember that the domain of $j$ is $x \leq 1$.

$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \quad x \leq 1 \\
&= 1 - \sqrt{(x^2 - 2x + 4) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) \qquad \text{since } x \leq 1 \\
&= x \ \checkmark
\end{aligned}$$

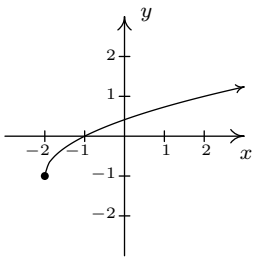
Checking $j \circ j^{-1}$, we get

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 + 2\sqrt{x-3} + 4 \\
&= 3 + x - 3 \\
&= x \ \checkmark
\end{aligned}$$

We can use what we know from Section 2.5 to graph $y = j^{-1}(x)$ on the same axes as $y = j(x)$ to get




2. We graph $y = k(x) = \sqrt{x+2} - 1$ using what we learned in Section 2.5 and see $k$ is one-to-one.




$y = k(x)$

We now try to find $k^{-1}$.

$$\begin{aligned} y &= k(x) \\ y &= \sqrt{x+2} - 1 \\ x &= \sqrt{y+2} - 1 \quad \text{switch } x \text{ and } y \\ x+1 &= \sqrt{y+2} \\ (x+1)^2 &= \left(\sqrt{y+2}\right)^2 \\ x^2 + 2x + 1 &= y + 2 \\ y &= x^2 + 2x - 1 \end{aligned}$$

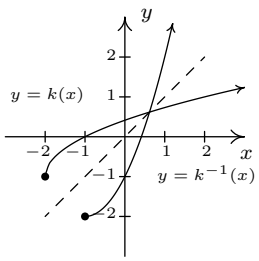
We have $k^{-1}(x) = x^2 + 2x - 1$. Based on our experience, we know something isn't quite right. We determined $k^{-1}$ is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem 6.2 tells us that the domain of $k^{-1}$ is the range of $k$. From the graph of $k$, we see that the range is $[-1, \infty)$, which means we restrict the domain of $k^{-1}$ to $x \geq -1$. We now check that this works in our compositions.

$$\begin{aligned} \left(k^{-1} \circ k\right)(x) &= k^{-1}(k(x)) \\ &= k^{-1}\left(\sqrt{x+2} - 1\right), \quad x \geq -2 \\ &= \left(\sqrt{x+2} - 1\right)^2 + 2\left(\sqrt{x+2} - 1\right) - 1 \\ &= \left(\sqrt{x+2}\right)^2 - 2\sqrt{x+2} + 1 + 2\sqrt{x+2} - 2 - 1 \\ &= x + 2 - 2 \\ &= x \ \checkmark \end{aligned}$$

and

$$\begin{aligned}
\left(k \circ k^{-1}\right)(x) &= k\left(x^2+2x-1\right) \quad x \geq -1 \\
&= \sqrt{(x^2+2x-1)+2}-1 \\
&= \sqrt{x^2+2x+1}-1 \\
&= \sqrt{(x+1)^2}-1 \\
&= |x+1|-1 \\
&= x+1-1 \qquad \text{since } x \geq -1 \\
&= x \ \checkmark
\end{aligned}$$

Graphically, everything checks out as well, provided that we remember the domain restriction on $k^{-1}$ means we take the right half of the parabola.




□

Our last example of the section gives an application of inverse functions.

EXAMPLE 6.2.4. Recall from Section 3.1 that the price-demand equation for the PortaBoy game system is $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$, where $x$ represents the number of systems sold weekly and $p$ is the price per system in dollars.

1. Explain why $p$ is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section 3.3 that the weekly profit $P$, in dollars, as a result of selling $x$ systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 3.3.3.

SOLUTION.

1. We leave to the reader to show the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, $p$ is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of $p^{-1}$ should match the range of $p$, which is $[1, 250]$, and as such, we restrict the domain of $p^{-1}$ to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function $p$ took as inputs the weekly sales and furnished the price per system as the output, $p^{-1}$ takes the price per system and returns the weekly sales as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in a week if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[7] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$. To understand what this means, recall that the original profit function $P$ gave us the weekly profit as a function of the weekly sales. The function $p^{-1}$ gives us the weekly sales as a function of the price. Hence, $P \circ p^{-1}$ takes as its input a price. The function $p^{-1}$ returns the weekly sales, which in turn is fed into $P$ to return the weekly profit. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the weekly profit (in dollars) as a function of the price per system, $x$, using the weekly sales $p^{-1}(x)$ as the 'middle man'.

4. We know from Section 3.3 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the $x$-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 3.4, $x = -\frac{b}{2a} = 165$. Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 3.3.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

[7] It is good review to actually do this!

### 6.2.1 EXERCISES

1. Show that the following functions are one-to-one and find the inverse. Check your answers algebraically and graphically. Verify the range of $f$ is the domain of $f^{-1}$ and vice-versa.

   (a) $f(x) = 6x - 2$
   (b) $f(x) = 5x - 3$
   (c) $f(x) = 1 - \dfrac{4+3x}{5}$
   (d) $f(x) = -\sqrt{x-5} + 2$
   (e) $f(x) = \sqrt{3x-1} + 5$
   (f) $f(x) = \sqrt[5]{3x-1}$
   (g) $f(x) = x^2 - 10x,\ x \geq 5$
   (h) $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
   (i) $f(x) = x^2 - 6x + 5,\ x \leq 3$
   (j) $f(x) = 4x^2 + 4x + 1,\ x < -1$
   (k) $f(x) = \dfrac{3}{4-x}$
   (l) $f(x) = \dfrac{x}{1-3x}$
   (m) $f(x) = \dfrac{2x-1}{3x+4}$
   (n) $f(x) = \dfrac{4x+2}{3x-6}$
   (o) $f(x) = \dfrac{-3x-2}{x+3}$

2. Show that the Fahrenheit to Celsius conversion function found in Exercise 3 in Section 3.1 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

3. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem 6.2 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$.

4. With the help of your classmates, find a formula for the inverse of the following.

   (a) $f(x) = ax + b,\ a \neq 0$
   (b) $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$
   (c) $f(x) = \frac{ax+b}{cx+d},\ a \neq 0,\ b \neq 0,\ c \neq 0,\ d \neq 0$
   (d) $f(x) = ax^2 + bx + c$ where $a \neq 0$, $x \geq -\frac{b}{2a}$.

5. Let $f(x) = \frac{2x}{x^2-1}$. Using the techniques in Section 5.2, graph $y = f(x)$. Verify $f$ is one-to-one on the interval $(-1, 1)$. Use the procedure outlined on Page 316 and your graphing calculator to find the formula for $f^{-1}(x)$. Note that since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

6. Suppose $f$ is an invertible function. Prove that if graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at all, they do so on the line $y = x$.

7. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

8. Let $f$ and $g$ be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

9. What graphical feature must a function $f$ possess for it to be its own inverse?

### 6.2.2 Answers

1. (a) $f^{-1}(x) = \dfrac{x+2}{6}$
   (b) $f^{-1}(x) = \dfrac{x+3}{5}$
   (c) $f^{-1}(x) = -\frac{5}{3}x + \frac{1}{3}$
   (d) $f^{-1}(x) = (x-2)^2 + 5,\ x \leq 2$
   (e) $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3},\ x \geq 5$
   (f) $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
   (g) $f^{-1}(x) = 5 + \sqrt{x+25}$
   (h) $f^{-1}(x) = -\sqrt{\frac{x+5}{3}} - 4$
   (i) $f^{-1}(x) = 3 - \sqrt{x+4}$
   (j) $f^{-1}(x) = -\frac{\sqrt{x}+1}{2},\ x > 1$
   (k) $f^{-1}(x) = \dfrac{4x-3}{x}$
   (l) $f^{-1}(x) = \dfrac{x}{3x+1}$
   (m) $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
   (n) $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
   (o) $f^{-1}(x) = \dfrac{-3x-2}{x+3}$

3. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

20 Function Composition 21 Inverse Functions

# Chapter 7

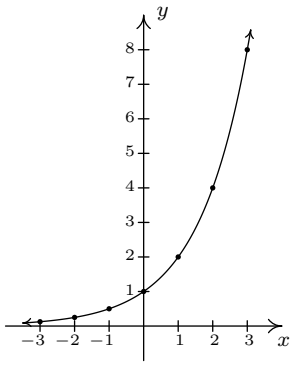
# Exponential and Logarithmic Functions

## 7.1 Introduction to Exponential and Logarithmic Functions

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most.[1] This section will introduce us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like $x^2$ or $x^{2/3}$, in other words, terms of the form $x^p$ where the base of the term, $x$, varies but the exponent of each term, $p$, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base $b$ is a constant and the exponent $x$ is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every College Algebra book we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect them in a pleasing fashion.

[1] Take a class in Differential Equations and you'll see why.

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-3$ | $2^{-3} = \frac{1}{8}$ | $\left(-3, \frac{1}{8}\right)$ |
| $-2$ | $2^{-2} = \frac{1}{4}$ | $\left(-2, \frac{1}{4}\right)$ |
| $-1$ | $2^{-1}$ | $\left(-1, \frac{1}{2}\right)$ |
| $0$ | $2^0 = 1$ | $(0, 1)$ |
| $1$ | $2^1 = 2$ | $(1, 2)$ |
| $2$ | $2^2 = 4$ | $(2, 4)$ |
| $3$ | $2^3 = 8$ | $(2, 8)$ |



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$ and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the $x$-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of $f$ is $(0, \infty)$. The graph of $f$ passes the Horizontal Line Test which means $f$ is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous[2] and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation. The number $\sqrt{3} = 1.73205\ldots$ is an irrational number[3] and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason[4] we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands

[2] Recall that this means there are no holes or other kinds of breaks in the graph.

[3] You can actually prove this by considering the polynomial $p(x) = x^2 - 3$ and showing it has no rational zeros by applying Theorem **??**.

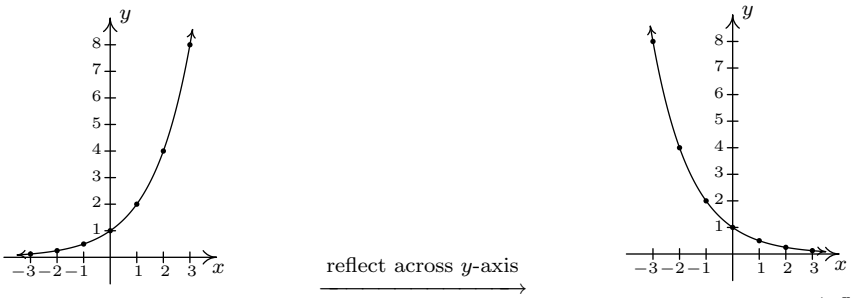
[4] This is where Calculus and continuity come into play.

to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.[5]

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases $b$ make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if $x$ is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and $0^0$ is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 7.1. A function of the form $f(x) = b^x$ where $b$ is a fixed real number, $b > 0$, $b \neq 1$ is called a **base $b$ exponential function**.

We leave it to the reader to verify[6] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Thinking back to Section 2.5, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the $y$-axis. As such, we have




$y = f(x) = 2^x$ $\qquad$ $y = g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of $g$ match that of $f$, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like $f$, $g$ is also one-to-one. Whereas $f$ is always increasing, $g$ is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of $g$. We summarize these observations, and more, in the following theorem whose proof ultimately requires Calculus.

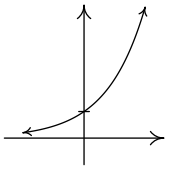
[5] Want more information? Look up "convergent sequences" on the Internet.

[6] Meaning, graph some more examples on your own.

THEOREM 7.1. **Properties of Exponential Functions:** Suppose $f(x) = b^x$.

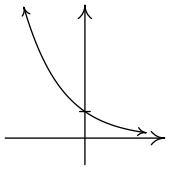
- The domain of $f$ is $(-\infty, \infty)$ and the range of $f$ is $(0, \infty)$.
- $(0, 1)$ is on the graph of $f$ and $y = 0$ is a horizontal asymptote to the graph of $f$.
- $f$ is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - $f$ is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of $f$ resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - $f$ is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of $f$ resembles:



$y = b^x$, $0 < b < 1$

[a] Recall that this means the graph of $f$ has no sharp turns or corners.

Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. We will more formally discuss the origins of this number in Section 7.5. For now, it is enough to know that since $e > 1$, $f(x) = e^x$ is an increasing exponential function. The following examples offer a glimpse as to the kind of real-world phenomena these functions can model.

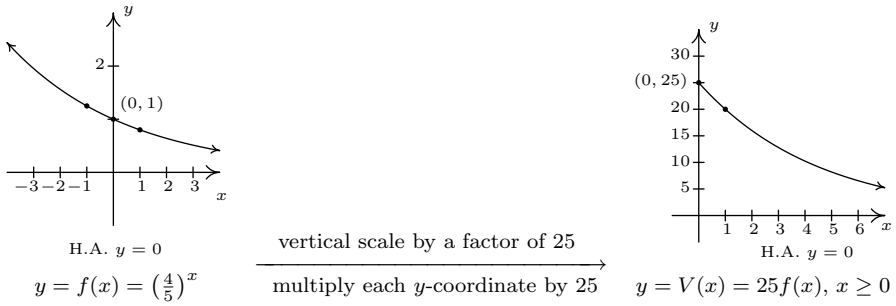
EXAMPLE 7.1.1. The value of a car can be modeled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

SOLUTION.

1. To find $V(0)$, we replace $x$ with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since $x$ represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of $\$25,000$. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was $\$25,000$.

2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the $y$-intercept $(0, 1)$ and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from $f$ by 25, in other words, $V(x) = 25f(x)$. In accordance with Theorem 2.5, this results in a vertical stretch by a factor of 25. We multiply all of the $y$ values in the graph by 25 (including the $y$ value of the horizontal asymptote) and obtain the points $\left(-1, \frac{125}{4}\right)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us. We have the result below.




3. We see from the graph of $V$ that its horizontal asymptote is $y = 0$. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0.

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 7.5. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations from Section 2.5.

Example 7.1.2. According to Netwon's Law of Cooling[7] the temperature of coffee $T$ (in degrees Fahrenheit) $t$ minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.

2. Sketch the graph of $y = T(t)$ using transformations.

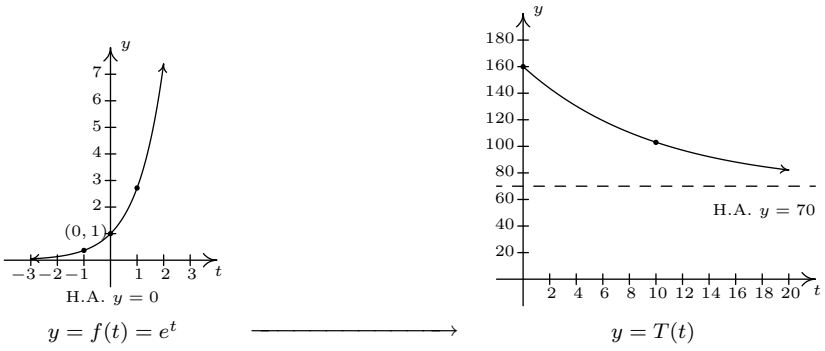
3. Find and interpret the horizontal asymptote of the graph.

[7] We will discuss this in greater detail in Section 7.5.

SOLUTION.

1. To find $T(0)$, we replace every occurrence of the independent variable $t$ with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of $f$ is an increasing exponential with $y$-intercept $(0, 1)$ and horizontal asymptote $y = 0$. The points $\left(-1, e^{-1}\right) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. Since the formula $T(t)$ looks rather complicated, we rewrite $T(t)$ in the form presented in Theorem 2.7 and use that result to track the changes to our three points and the horizontal asymptote. We have $T(t) = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Multiplication of the input to $f$, $t$, by $-0.1$ results in a horizontal expansion by a factor of 10 as well as a reflection about the $y$-axis. We divide each of the $x$ values of our points by $-0.1$ (which amounts to multiplying them by $-10$) to obtain $\left(10, e^{-1}\right)$, $(0, 1)$, and $(-10, e)$. Since none of these changes affected the $y$ values, the horizontal asymptote remains $y = 0$. Next, we see that the output from $f$ is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the $y$-coordinates by 90 to obtain $\left(10, 90e^{-1}\right)$, $(0, 90)$, and $(-10, 90e)$. We also multiply the $y$ value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the $y$-coordinates, which shifts the graph upwards to obtain $\left(10, 90e^{-1} + 70\right) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of $f$ and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is below.




3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically and to review some of the 'number sense' developed in Chapter 5. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big } (-)}$. Since $e > 1$, $e^{\text{very big } (-)} = \frac{1}{e^{\text{very big } (+)}} \approx \frac{1}{\text{very big } (+)} \approx$ very small $(+)$. The larger $t$ becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70 +$ very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature. □

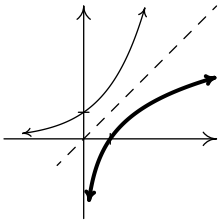
As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 7.2. The inverse of the exponential function $f(x) = b^x$ is called the **base $b$ logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ The expression $\log_b(x)$ is read 'log base $b$ of $x$.'

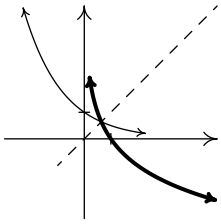
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 7.3. The **common logarithm** of a real number $x$ is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number $x$ is $\log_e(x)$ and is usually written $\ln(x)$.

Since logs are defined as the inverses of exponential functions, we can use Theorems 6.2 and 6.3 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) = b^x$ for the different cases of $b$, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of $f$ across the line $y = x$ as shown below. The $y$-intercept $(0, 1)$ on the graph of $f$ corresponds to an $x$-intercept of $(1, 0)$ on the graph of $f^{-1}$. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$
$\boldsymbol{y = \log_b(x)}$, $b > 1$



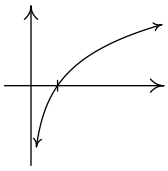
$y = b^x$, $0 < b < 1$
$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get $x$. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

THEOREM 7.2. **Properties of Logarithmic Functions:** Suppose $f(x) = \log_b(x)$.

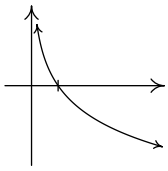
- The domain of $f$ is $(0, \infty)$ and the range of $f$ is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of $f$ and $x = 0$ is a vertical asymptote of the graph of $f$.
- $f$ is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on $b$ to obtain $c$.
- $\log_b(b^x) = x$ for all $x$ and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - $f$ is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of $f$ resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - $f$ is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of $f$ resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 7.2 is a consequence of Theorems 6.2 and 6.3. However, it is worth the reader's time to understand Theorem 7.2 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us $-1$. In other words, we are looking for $x$ that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 7.2 at this level instead of just merely memorizing the facts.

EXAMPLE 7.1.3. Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\frac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

Solution.

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.
2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.
3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.
4. First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on $e$ to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.
5. Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.
6. We can use Theorem 7.2 directly to simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.
7. We note that we cannot apply Theorem 7.2 directly to $117^{-\log_{117}(6)}$. (Why not?) We use a property of exponents to rewrite $117^{-\log_{117}(6)}$ as $\dfrac{1}{117^{\log_{117}(6)}}$. At this point, we can apply Theorem 7.2 to get $117^{\log_{117}(6)} = 6$ and thus $117^{-\log_{117}(6)} = \dfrac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$. It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$. □

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument[8] of the log must be strictly positive.

[8] See page 87 if you've forgotten what this term means.

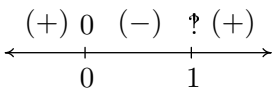
EXAMPLE 7.1.4. Find the domain of the following functions. Check your answers graphically using the calculator.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

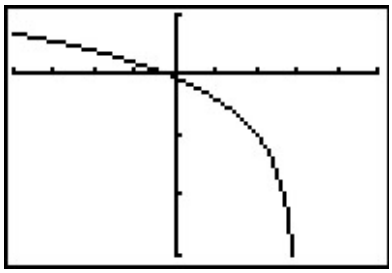
SOLUTION.

1. We set $3-x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph from the calculator below verifies this. Note that we could have graphed $f$ using transformations. Taking a cue from Theorem 2.7, we rewrite $f(x) = 2\log_{10}(-x+3) - 1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x + 3) - 1$, Theorem 2.7 tells us that to obtain the destinations of these points, we first subtract 3 from the $x$-coordinates (shifting the graph left 3 units), then divide (multiply) by the $x$-coordinates by $-1$ (causing a reflection across the $y$-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multplying by $-1$ gives us the vertical asymptote $x = 3$. Next, we multiply the $y$-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the $y$-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.

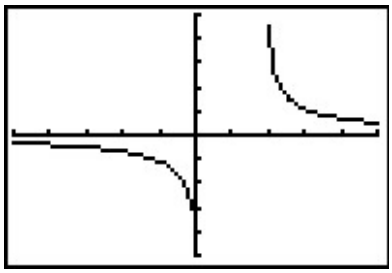
2. To find the domain of $g$, we set $\frac{x}{x-1} > 0$ and use a sign diagram to solve this inequality. We define $r(x) = \frac{x}{x-1}$ find its domain to be $r$ is $(-\infty, 1) \cup (1, \infty)$. Setting $r(x) = 0$ gives $x = 0$.




We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of $g$. The graph of $y = g(x)$ confirms this. We can tell from the graph of $g$ that it is not the result of Section 2.5 transformations being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behavior of $g$. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter 5 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.



$y = f(x) = 2\log(3 - x) - 1$



$y = g(x) = \ln\left(\dfrac{x}{x-1}\right)$

□

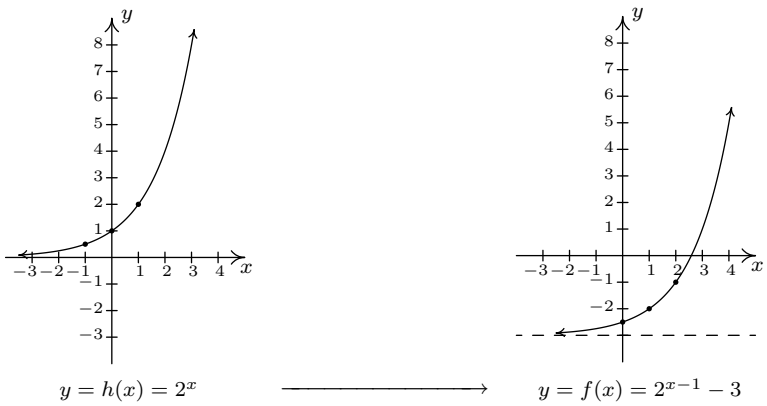
While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

Example 7.1.5. Let $f(x) = 2^{x-1} - 3$.

1. Graph $f$ using transformations and state the domain and range of $f$.
2. Explain why $f$ is invertible and find a formula for $f^{-1}(x)$.
3. Graph $f^{-1}$ using transformations and state the domain and range of $f^{-1}$.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all $x$ in the domain of $f$ and $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the domain of $f^{-1}$.
5. Graph $f$ and $f^{-1}$ on the same set of axes and check the symmetry about the line $y = x$.

Solution.

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x - 1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ on the graph of $g$ along with the horizontal asymptote $y = 0$ to track through the transformations. By Theorem 2.7 we first add 1 to the $x$-coordinates of the points on the graph of $g$ (shifting $g$ to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1, 1)$ and $(2, 2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the $y$-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1, -2)$ and $(2, -1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of $g$, we get the graph below. We see that the domain of $f$ is the same as $g$, namely $(-\infty, \infty)$, but that the range of $f$ is $(-3, \infty)$.




2. The graph of $f$ passes the Horizontal Line Test so $f$ is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the $x$ and $y$, then proceed to solve for $y$. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for $f^{-1}$ from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find $f$ takes an input $x$ and applies the steps

   (a) subtract 1
   (b) put as an exponent on 2
   (c) subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, $f^{-1}$ should

   (a) add 3
   (b) take the logarithm base 2
   (c) add 1

   In symbols, $f^{-1}(x) = \log_2(x + 3) + 1$.

3. To graph $f^{-1}(x) = \log_2(x + 3) + 1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of $j$ along with the vertical asymptote $x = 0$ through the transformations using Theorem 2.7. Since $f^{-1}(x) = j(x + 3) + 1$, we first subtract 3 from each of the $x$ values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, $(-2, 0)$ and $(-1, 1)$ with a vertical asymptote $x = -3$. Next, we add 1 to the $y$ values on the graph and get $\left(-\frac{5}{2}, 0\right)$, $(-2, 1)$ and $(-1, 2)$. If you are experiencing *déjà vu*, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of $f^{-1}$ is $(-3, \infty)$, which matches the range of $f$, and the range of $f^{-1}$ is $(-\infty, \infty)$, which matches the domain of $f$.

4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3) + 1$ satisfy the composition requirement for inverses. For all real numbers $x$,

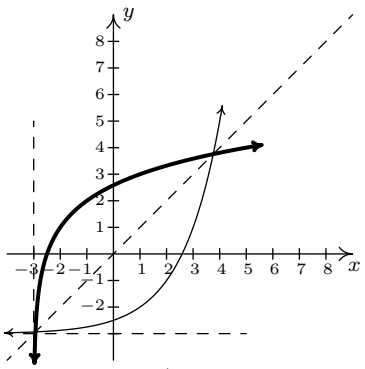
$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1} - 3\right) \\
&= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\
&= \log_2\left(2^{x-1}\right) + 1 \\
&= (x-1) + 1 && \text{Since } \log_2\left(2^u\right) = u \text{ for all real numbers } u \\
&= x \ \checkmark
\end{aligned}$$

For all real numbers $x > -3$, we have[9]

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3) + 1\right) \\
&= 2^{(\log_2(x+3)+1)-1} - 3 \\
&= 2^{\log_2(x+3)} - 3 \\
&= (x+3) - 3 && \text{Since } 2^{\log_2(u)} = u \text{ for all real numbers } u > 0 \\
&= x \ \checkmark
\end{aligned}$$

[9] Pay attention - can you spot in which step below we need $x > -3$?

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$.




$y = f(x) = 2^{x-1} - 3$

$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

□

### 7.1.1 Exercises

1. Evaluate the expression.

   (a) $\log_3(27)$
   (b) $\log_6(216)$
   (c) $\log_2(32)$
   (d) $\log_6\left(\frac{1}{36}\right)$
   (e) $\log_8(4)$
   (f) $\log_{36}(216)$
   (g) $\log_{\frac{1}{5}}(625)$
   (h) $\log_{\frac{1}{6}}(216)$
   (i) $\log_{36}(36)$
   (j) $\log\left(\frac{1}{1000000}\right)$
   (k) $\log(0.01)$
   (l) $\ln\left(e^3\right)$
   (m) $\log_4(8)$
   (n) $\log_6(1)$
   (o) $\log_{13}\left(\sqrt{13}\right)$
   (p) $\log_{36}\left(\sqrt[4]{36}\right)$
   (q) $7^{\log_7(3)}$
   (r) $36^{\log_{36}(216)}$
   (s) $\log_{36}\left(36^{216}\right)$
   (t) $\ln\left(e^5\right)$
   (u) $\log\left(\sqrt[9]{10^{11}}\right)$
   (v) $\log\left(\sqrt[3]{10^5}\right)$
   (w) $\ln\left(\frac{1}{\sqrt{e}}\right)$
   (x) $\log_5\left(3^{\log_3(5)}\right)$
   (y) $\log\left(e^{\ln(100)}\right)$

2. Find the domain of the function.

   (a) $f(x) = \ln(x^2+1)$
   (b) $f(x) = \log_7(4x+8)$
   (c) $f(x) = \ln(4x-20)$
   (d) $f(x) = \log\left(x^2+9x+18\right)$
   (e) $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$
   (f) $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$
   (g) $f(x) = \ln(7-x) + \ln(x-4)$
   (h) $f(x) = \ln(4x-20) + \ln\left(x^2+9x+18\right)$
   (i) $f(x) = \log\left(x^2+x+1\right)$
   (j) $f(x) = \sqrt[4]{\log_4(x)}$
   (k) $f(x) = \log_9(|x+3|-4)$
   (l) $f(x) = \ln(\sqrt{x-4}-3)$
   (m) $f(x) = \dfrac{1}{3-\log_5(x)}$
   (n) $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$
   (o) $f(x) = \ln(-2x^3-x^2+13x-6)$

3. For each function given below, find its inverse from the 'procedural perspective' discussed in Example 7.1.5 and graph the function and its inverse on the same set of axes.

   (a) $f(x) = 3^{x+2} - 4$
   (b) $f(x) = \log_4(x-1)$
   (c) $f(x) = -2^{-x} + 1$
   (d) $f(x) = 5\log(x) - 2$

4. Show that $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0,\ b \neq 1$.

5. (Crazy bonus question) Without using your calculator, determine which is larger: $e^\pi$ or $\pi^e$.

6. (The Logarithmic Scales) There are three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay close attention to the subtle differences.

(a) Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[10] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where $x$ is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

i. Show that $M(0.001) = 0$.

ii. Compute $M(80,000)$.

iii. Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.

iv. Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

(b) While the decibel scale can be used in many disciplines,[11] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound.[12] The Sound Intensity Level $L$ (measured in decibels) of a sound intensity $I$ (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares $I$ to baseline: $10^{-12}\frac{W}{m^2}$ is the threshold of human hearing.

i. Compute $L(10^{-6})$.

ii. Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity $I$ is needed to produce this level?

iii. Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

[10] Rock-solid, perhaps?

[11] See this webpage for more information.

[12] As of the writing of this exercise, the Wikipedia page given here states that it may not meet the "general notability guideline" nor does it cite any references or sources. I find this odd because it is this very usage of the decibel scale which shows up in every College Algebra book I have read. Perhaps those other books have been wrong all along and we're just blindly following tradition.

(c) The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

i. The hydrogen ion concentration of pure water is $[\text{H}^+] = 10^{-7}$. Find its pH.

ii. Find the pH of a solution with $[\text{H}^+] = 6.3 \times 10^{-13}$.

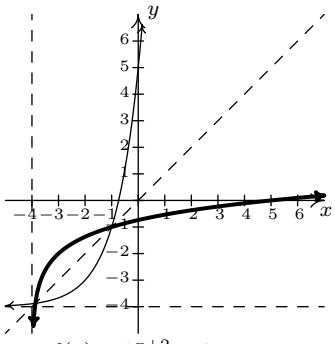
iii. The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

### 7.1.2 ANSWERS

1. (a) $\log_3(27) = 3$
   (b) $\log_6(216) = 3$
   (c) $\log_2(32) = 5$
   (d) $\log_6\left(\frac{1}{36}\right) = -2$
   (e) $\log_8(4) = \frac{2}{3}$
   (f) $\log_{36}(216) = \frac{3}{2}$
   (g) $\log_{\frac{1}{5}}(625) = -4$
   (h) $\log_{\frac{1}{6}}(216) = -3$
   (i) $\log_{36}(36) = 1$
   (j) $\log \frac{1}{1000000} = -6$
   (k) $\log(0.01) = -2$
   (l) $\ln\left(e^3\right) = 3$
   (m) $\log_4(8) = \frac{3}{2}$
   (n) $\log_6(1) = 0$
   (o) $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
   (p) $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
   (q) $7^{\log_7(3)} = 3$
   (r) $36^{\log_{36}(216)} = 216$
   (s) $\log_{36}\left(36^{216}\right) = 216$
   (t) $\ln(e^5) = 5$
   (u) $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
   (v) $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
   (w) $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
   (x) $\log_5\left(3^{\log_3 5}\right) = 1$
   (y) $\log\left(e^{\ln(100)}\right) = 2$

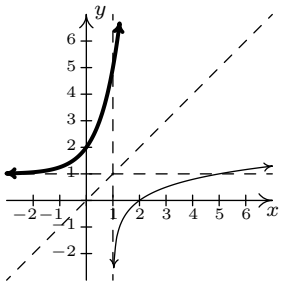
2. (a) $(-\infty, \infty)$
   (b) $(-2, \infty)$
   (c) $(5, \infty)$
   (d) $(-\infty, -6) \cup (-3, \infty)$
   (e) $(-2, -1) \cup (1, \infty)$
   (f) $(-6, -3) \cup (5, \infty)$
   (g) $(4, 7)$
   (h) $(5, \infty)$
   (i) $(-\infty, \infty)$
   (j) $[1, \infty)$
   (k) $(-\infty, -7) \cup (1, \infty)$
   (l) $(13, \infty)$
   (m) $(0, 125) \cup (125, \infty)$
   (n) No domain
   (o) $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

3. (a) $f(x) = 3^{x+2} - 4$
   $f^{-1}(x) = \log_3(x + 4) - 2$



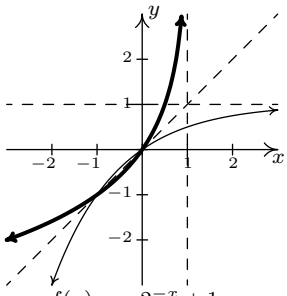

$y = f(x) = 3^{x+2} - 4$
$\boldsymbol{y = f^{-1}(x) = \log_3(x + 4) - 2}$

(b) $f(x) = \log_4(x - 1)$
$f^{-1}(x) = 4^x + 1$



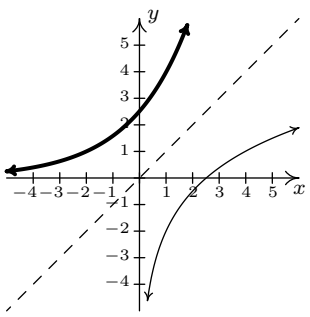

$y = f(x) = \log_4(x - 1)$
$\boldsymbol{y = f^{-1}(x) = 4^x + 1}$

(c) $f(x) = -2^{-x} + 1$
$f^{-1}(x) = -\log_2(1-x)$




$y = f(x) = -2^{-x} + 1$
$\boldsymbol{y = f^{-1}(x) = -\log_2(1-x)}$

(d) $f(x) = 5\log(x) - 2$
$f^{-1}(x) = 10^{\frac{x+2}{5}}$




$y = f(x) = 5\log(x) - 2$
$\boldsymbol{y = f^{-1}(x) = 10^{\frac{x+2}{5}}}$

6. (a) i. $M(0.001) = \log\left(\dfrac{0.001}{0.001}\right) = \log(1) = 0.$

ii. $M(80,000) = \log\left(\dfrac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9.$

(b) i. $L(10^{-6}) = 60$ decibels.

ii. $I = 10^{-.5} \approx 0.316$ watts per square meter.

iii. Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

(c) i. The pH of pure water is 7.

ii. If $[\mathrm{H}^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.

iii. $[\mathrm{H}^+] = 10^{-0.7} \approx .1995$ moles per liter.

## 7.2 PROPERTIES OF LOGARITHMS

In Section 7.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge rule in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 7.2 to remind us of the definition of a logarithm as the inverse of an exponential function.

THEOREM 7.3. **(Inverse Properties of Exponential and Log Functions)** Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b(b^x) = x$ for all $x$ and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out in more detail what it means for exponential and logarithmic functions to be one-to-one.

THEOREM 7.4. **(One-to-one Properties of Exponential and Log Functions)** Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then $f$ and $g$ are one-to-one. In other words:

- $b^u = b^w$ if and only if $u = w$ for all real numbers $u$ and $w$.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base $b$ is restricted to $b > 0$, $b \neq 1$. An added benefit of this restriction is that it eliminates the pathologies discussed in Section **??** when, for example, we simplified $\left(x^{2/3}\right)^{3/2}$ and obtained $|x|$ instead of what we had expected from the arithmetic in the exponents, $x^1 = x$.

Theorem 7.5. **(Algebraic Properties of Exponential Functions)** Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let $u$ and $w$ be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 7.5 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 7.6. **(Algebraic Properties of Logarithm Functions)** Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 7.6 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of $b^x$, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly. From a purely functional approach, we can see the properties in Theorem 7.6 as an example of how inverse functions interchange the roles of inputs in outputs. For instance, the Product Rule for exponential functions given in Theorem 7.5, $f(u+w) = f(u)f(w)$, says that adding inputs results in multiplying outputs. Hence, whatever $f^{-1}$ is, it must take the products of outputs from $f$ and return them to the sum of their respective inputs. Since the outputs from $f$ are the inputs to $f^{-1}$ and vice-versa, we have that that $f^{-1}$ must take products of its inputs to the sum of their respective outputs. This is precisely what the Product Rule for Logarithmic functions states in Theorem 7.6: $g(uw) = g(u)+g(w)$. The reader is encouraged to view the remaining properties listed in Theorem 7.6 similarly. The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 7.6 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and

powers inside the log as factors outside the log. While it is the opposite process, which we will practice later, that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

EXAMPLE 7.2.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\frac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\frac{3}{ex}\right)^2$
4. $\log\sqrt[3]{\frac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2-4\right)$

SOLUTION.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u=8$ and $w=x$ and simplify.

$$\begin{aligned}\log_2\left(\frac{8}{x}\right) &= \log_2(8)-\log_2(x) && \text{Quotient Rule}\\ &= 3-\log_2(x) && \text{Since } 2^3=8\\ &= -\log_2(x)+3\end{aligned}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the $x^2$) and a product. In order to use the Product Rule, the *entire* quantity inside the logarithm must be raised to the same exponent. Since the exponent 2 applies only to the $x$, we first apply the Product Rule with $u=10$ and $w=x^2$. Once we get the $x^2$ by itself inside the log, we may apply the Power Rule with $u=x$ and $w=2$ and simplify.

$$\begin{aligned}\log_{0.1}\left(10x^2\right) &= \log_{0.1}(10)+\log_{0.1}\left(x^2\right) && \text{Product Rule}\\ &= \log_{0.1}(10)+2\log_{0.1}(x) && \text{Power Rule}\\ &= -1+2\log_{0.1}(x) && \text{Since } (0.1)^{-1}=10\\ &= 2\log_{0.1}(x)-1\end{aligned}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{ex}$ and $w = 2$. Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3) - \ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base $e$.

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule}\\
&= 2\left[\ln(3) - \ln(ex)\right] && \text{Quotient Rule}\\
&= 2\ln(3) - 2\ln(ex)\\
&= 2\ln(3) - 2\left[\ln(e) + \ln(x)\right] && \text{Product Rule}\\
&= 2\ln(3) - 2\ln(e) - 2\ln(x)\\
&= 2\ln(3) - 2 - 2\ln(x) && \text{Since } e^1 = e\\
&= -2\ln(x) + 2\ln(3) - 2
\end{aligned}$$

4. In Theorem 7.6, there is no mention of how to deal with radicals. However, thinking back to Definition **??**, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power

Rule[1], and we keep in mind that the common log is log base 10.

$$\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3} \\
&= \tfrac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule} \\
&= \tfrac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] && \text{Quotient Rule} \\
&= \tfrac{1}{3}\log\left(100x^2\right) - \tfrac{1}{3}\log\left(yz^5\right) \\
&= \tfrac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \tfrac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] && \text{Product Rule} \\
&= \tfrac{1}{3}\log(100) + \tfrac{1}{3}\log\left(x^2\right) - \tfrac{1}{3}\log(y) - \tfrac{1}{3}\log\left(z^5\right) \\
&= \tfrac{1}{3}\log(100) + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Power Rule} \\
&= \tfrac{2}{3} + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Since } 10^2 = 100 \\
&= \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) + \tfrac{2}{3}
\end{aligned}$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2 - 4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2 - 4 = (x + 2)(x - 2)$ thereby introducing a product which gives us license to use the Product Rule.

$$\begin{aligned}
\log_{117}\left(x^2 - 4\right) &= \log_{117}\left[(x + 2)(x - 2)\right] && \text{Factor} \\
&= \log_{117}(x + 2) + \log_{117}(x - 2) && \text{Product Rule}
\end{aligned}$$

□

A couple of remarks about Example 7.2.1 are in order. First, while not explicitly stated in the above example, a general rule of thumb to determine which log property to apply first to a complicated problem is 'reverse order of operations.' For example, if we were to substitute a number for $x$ into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the $x$, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. In a

[1] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of $u$ and which is playing the role of $w$ as we apply each property.

multi-step problem, this rule can give the required guidance on which log property to apply at each step. The reader is encouraged to look through the solutions to Example 7.2.1 to see this rule in action. Second, while we were instructed to assume when necessary that all quantities represented positive real numbers, the authors would be committing a sin of omission if we failed to point out that, for instance, the functions $f(x) = \log_{117}\left(x^2 - 4\right)$ and $g(x) = \log_{117}(x + 2) + \log_{117}(x - 2)$ have different domains, and, hence, are different functions. We leave it to the reader to verify the domain of $f$ is $(-\infty, -2) \cup (2, \infty)$ whereas the domain of $g$ is $(2, \infty)$. In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so. One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2 - 4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten[2] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 7.2.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x - 1) - \log_3(x + 1)$
2. $\log(x) + 2\log(y) - \log(z)$
3. $4\log_2(x) + 3$
4. $-\ln(x) - \frac{1}{2}$

Solution. Whereas in Example 7.2.1 we read the properties in Theorem 7.6 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1)-\log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.
2. In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x) + 2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it all together, we have

$$\begin{aligned} \log(x) + 2\log(y) - \log(z) &= \log(x) + \log\left(y^2\right) - \log(z) && \text{Power Rule} \\ &= \log\left(xy^2\right) - \log(z) && \text{Product Rule} \\ &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule} \end{aligned}$$

3. We can certainly get started rewriting $4\log_2(x) + 3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 7.3, we know $3 = \log_2\left(2^3\right)$, so we get

[2]The authors relish the irony involved in writing what follows.

$$\begin{aligned} 4\log_2(x)+3 &= \log_2\left(x^4\right)+3 && \text{Power Rule} \\ &= \log_2\left(x^4\right)+\log_2\left(2^3\right) && \text{Since } 3=\log_2\left(2^3\right) \\ &= \log_2\left(x^4\right)+\log_2(8) \\ &= \log_2\left(8x^4\right) && \text{Product Rule} \end{aligned}$$

4. To get started with $-\ln(x)-\frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x)=\ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 7.3 gives us $\frac{1}{2}=\ln\left(e^{1/2}\right)=\ln\left(\sqrt{e}\right)$. We have

$$\begin{aligned} -\ln(x)-\tfrac{1}{2} &= (-1)\ln(x)-\tfrac{1}{2} \\ &= \ln\left(x^{-1}\right)-\tfrac{1}{2} && \text{Power Rule} \\ &= \ln\left(x^{-1}\right)-\ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2}=\ln\left(e^{1/2}\right) \\ &= \ln\left(x^{-1}\right)-\ln\left(\sqrt{e}\right) \\ &= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\ &= \ln\left(\frac{1}{x\sqrt{e}}\right) \end{aligned}$$

□

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x)=\log_3(x-1)-\log_3(x+1)$ is $(1,\infty)$ but the domain of $g(x)=\log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty,-1)\cup(1,\infty)$. We will need to keep this in mind when we solve equations involving logarithms in Section 7.4 - it is precisely for this reason we will have to check for extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 7.7. **(Change of Base)** Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers $x$.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

The proofs of the Change of Base formulas are a result of the other properties studied in this section. If we start with $b^{x \log_b(a)}$ and use the Power Rule in the exponent to rewrite $x \log_b(a)$ as $\log_b(a^x)$ and then apply one of the Inverse Properties in Theorem 7.3, we get

$$b^{x \log_b(a)} = b^{\log_b(a^x)} = a^x,$$

as required. To verify the logarithmic form of the property, we also use the Power Rule and an Inverse Property. We note that

$$\log_a(x) \cdot \log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x),$$

and we get the result by dividing through by $\log_b(a)$. Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$. While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.[3] What Theorem 7.7 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base - be it 10, $e$, 42, or 117. Your Calculus teacher will have more to say about this when the time comes.

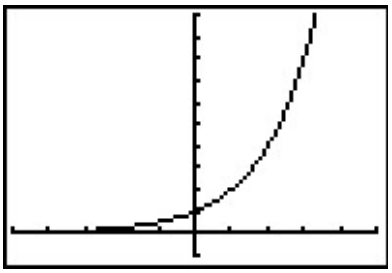
Example 7.2.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a calculator, as appropriate.

[3] The authors feel so strongly about showing students that every property of logarithms comes from and corresponds to a property of exponents that we have broken tradition with the vast majority of other authors in this field. This isn't the first time this happened, and it certainly won't be the last.

1. $3^2$ to base 10
2. $2^x$ to base $e$
3. $\log_4(5)$ to base $e$
4. $\ln(x)$ to base 10

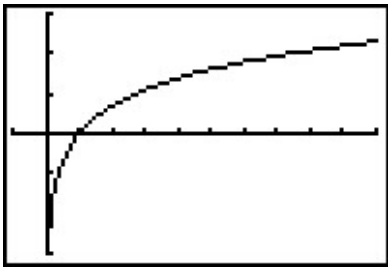
SOLUTION.

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.
2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ and $g(x) = e^{x\ln(2)}$. Their graphs are indistinguishable which provides evidence that they are the same function.



$y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The calculator confirms this.[4]
4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.



$y = f(x) = \ln(x)$ and $y = g(x) = \frac{\log(x)}{\log(e)}$

[4]Which means if it is lying to us about the first answer it gave us, at least it is being consistent.

### 7.2.1 Exercises

1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

   (a) $\ln(x^3y^2)$
   (b) $\log_2\left(\dfrac{128}{x^2+4}\right)$
   (c) $\log_5\left(\dfrac{z}{25}\right)^3$
   (d) $\log(1.23\times 10^{37})$
   (e) $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
   (f) $\log_5\left(x^2-25\right)$
   (g) $\log_{\sqrt{2}}\left(4x^3\right)$
   (h) $\log_{\frac{1}{3}}(9x(y^3-8))$
   (i) $\log\left(1000x^3y^5\right)$
   (j) $\log_3\left(\dfrac{x^2}{81y^4}\right)$
   (k) $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$
   (l) $\log_6\left(\dfrac{216}{x^3y}\right)^4$
   (m) $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

2. Use the properties of logarithms to write the following as a single logarithm.

   (a) $4\ln(x)+2\ln(y)$
   (b) $3-\log(x)$
   (c) $\log_2(x)+\log_2(y)-\log_2(z)$
   (d) $\log_3(x)-2\log_3(y)$
   (e) $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
   (f) $2\ln(x)-3\ln(y)-4\ln(z)$
   (g) $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
   (h) $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
   (i) $\log_2(x)+\log_{\frac{1}{2}}(x-1)$
   (j) $\log_2(x)+\log_4(x-1)$
   (k) $\log_5(x)-3$
   (l) $\log_7(x)+\log_7(x-3)-2$
   (m) $\ln(x)+\frac{1}{2}$

3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base.

   (a) $7^{x-1}$ to base $e$
   (b) $\log_3(x+2)$ to base 10
   (c) $\left(\dfrac{2}{3}\right)^x$ to base $e$
   (d) $\log(x^2+1)$ to base $e$

4. Use the appropriate change of base formula to approximate the following logarithms.

   (a) $\log_3(12)$
   (b) $\log_5(80)$
   (c) $\log_6(72)$
   (d) $\log_4\left(\dfrac{1}{10}\right)$
   (e) $\log_{\frac{3}{5}}(1000)$
   (f) $\log_{\frac{2}{3}}(50)$

5. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

6. Prove the Quotient Rule and Power Rule for Logarithms.

7. Give numerical examples to show that, in general,

   (a) $\log_b(x+y) \neq \log_b(x) + \log_b(y)$
   (b) $\log_b(x-y) \neq \log_b(x) - \log_b(y)$
   (c) $\log_b\left(\dfrac{x}{y}\right) \neq \dfrac{\log_b(x)}{\log_b(y)}$

8. The Henderson-Hasselbalch Equation: Suppose $HA$ represents a weak acid. Then we have a reversible chemical reaction

$$HA \rightleftharpoons H^+ + A^-.$$

   The acid disassociation constant, $K_a$, is given by

$$K_\alpha = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

   where the square brackets denote the concentrations just as they did in Exercise 6c in Section 7.1. The symbol $\mathrm{p}K_a$ is defined similarly to pH in that $\mathrm{p}K_a = -\log(K_a)$. Using the definition of pH from Exercise 6c and the properties of logarithms, derive the Henderson-Hasselbalch Equation which states

$$\mathrm{pH} = \mathrm{p}K_a + \log\frac{[A^-]}{[HA]}$$

9. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

10. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

### 7.2.2 Answers

1. (a) $3\ln(x) + 2\ln(y)$
   (b) $7 - \log_2(x^2 + 4)$
   (c) $3\log_5(z) - 6$
   (d) $\log(1.23) + 37$
   (e) $\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$
   (f) $\log_5(x - 5) + \log_5(x + 5)$
   (g) $3\log_{\sqrt{2}}(x) + 4$
   (h) $-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y-2) + \log_{\frac{1}{3}}(y^2+2y+4)$
   (i) $3 + 3\log(x) + 5\log(y)$
   (j) $2\log_3(x) - 4 - 4\log_3(y)$
   (k) $\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$
   (l) $12 - 12\log_6(x) - 4\log_6(y)$
   (m) $\frac{1}{3}\ln(x) - \ln(10) - \frac{1}{2}\ln(y) - \frac{1}{2}\ln(z)$

2. (a) $\ln(x^4y^2)$
   (b) $\log\left(\dfrac{1000}{x}\right)$
   (c) $\log_2\left(\dfrac{xy}{z}\right)$
   (d) $\log_3\left(\dfrac{x}{y^2}\right)$
   (e) $\log_3\left(\dfrac{\sqrt{x}}{y^2z}\right)$
   (f) $\ln\left(\dfrac{x^2}{y^3z^4}\right)$
   (g) $\log\left(\dfrac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
   (h) $\ln\left(\sqrt[3]{\dfrac{z}{xy}}\right)$
   (i) $\log_2\left(\dfrac{x}{x-1}\right)$
   (j) $\log_2\left(x\sqrt{x-1}\right)$
   (k) $\log_5\left(\dfrac{x}{125}\right)$
   (l) $\log_7\left(\dfrac{x(x-3)}{49}\right)$
   (m) $\ln\left(x\sqrt{e}\right)$

3. (a) $7^{x-1} = e^{(x-1)\ln(7)}$
   (b) $\log_3(x+2) = \dfrac{\log(x+2)}{\log(3)}$
   (c) $\left(\dfrac{2}{3}\right)^x = e^{x\ln(\frac{2}{3})}$
   (d) $\log(x^2+1) = \dfrac{\ln(x^2+1)}{\ln(10)}$

4. (a) $\log_3(12) \approx 2.26186$
   (b) $\log_5(80) \approx 2.72271$
   (c) $\log_6(72) \approx 2.38685$
   (d) $\log_4\left(\dfrac{1}{10}\right) \approx -1.66096$
   (e) $\log_{\frac{3}{5}}(1000) \approx -13.52273$
   (f) $\log_{\frac{2}{3}}(50) \approx -9.64824$

## 7.3 EXPONENTIAL EQUATIONS AND INEQUALITIES

In this section we will develop techniques for solving equations involving exponential functions. Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 7.4, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution. Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 7.3 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$. This makes sense because, after all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 7.7, to give us something more calculator friendly,[1] say $\log_2(129) = \frac{\ln(129)}{\ln(2)}$. Another way to arrive at this answer is as follows

$$\begin{aligned} 2^x &= 129 && \\ \ln(2^x) &= \ln(129) && \text{Take the natural log of both sides.} \\ x\ln(2) &= \ln(129) && \text{Power Rule} \\ x &= \frac{\ln(129)}{\ln(2)} && \end{aligned}$$

'Taking the natural log' of both sides is akin to squaring both sides: since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[2] Also note that we treat $\ln(2)$ as any other non-zero real number and divide it through[3] to isolate the variable $x$. We summarize below the two common ways to solve exponential equations, motivated by our examples.

**Steps for Solving an Equation involving Exponential Functions**

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

EXAMPLE 7.3.1. Solve the following equations. Check your answer graphically using a calculator.

[1] You can use natural logs or common logs. We choose natural logs. (In Calculus, you'll learn these are the most 'mathy' of the logarithms.)

[2] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 7.4.

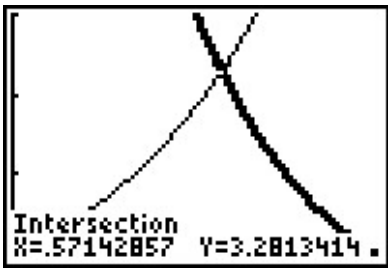
[3] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{}$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

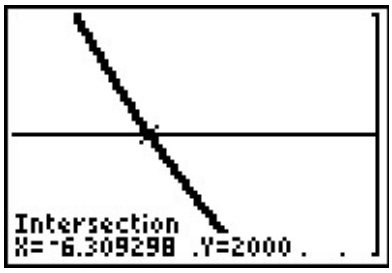
Solution.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$. Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$. To check graphically, we set $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.5714$.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$. Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$. Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ to get $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$. On the calculator, we graph $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ and find that they intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.3093$.




$y = f(x) = 2^{3x}$ and
$\boldsymbol{y = g(x) = 16^{1-x}}$



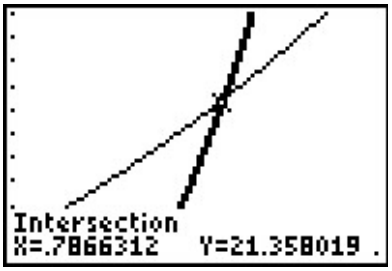

$y = f(x) = 2000$ and
$\boldsymbol{y = g(x) = 1000 \cdot 3^{-0.1x}}$

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$. Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$. The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants. The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation and as such we gather all of the terms with $x$ on one side, and the constants on the other. We then divide both sides by the coefficient of $x$, which we obtain by factoring.

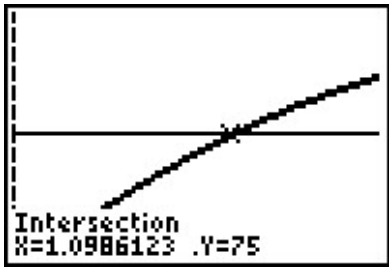
$$\begin{aligned}
(x+2)\ln(3) &= 2x\ln(7) \\
x\ln(3)+2\ln(3) &= 2x\ln(7) \\
2\ln(3) &= 2x\ln(7)-x\ln(3) \\
2\ln(3) &= x(2\ln(7)-\ln(3)) \quad \text{Factor.} \\
x &= \tfrac{2\ln(3)}{2\ln(7)-\ln(3)}
\end{aligned}$$

Graphing $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ on the calculator, we see that these two graphs intersect at $x = \frac{2\ln(3)}{2\ln(7)-\ln(3)} \approx 0.7866$.

4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential. To that end, we clear denominators and get $75\left(1+3e^{-2t}\right) = 100$. From this we get $75 + 225e^{-2t} = 100$, which leads to $225e^{-2t} = 25$, and finally, $e^{-2t} = \frac{1}{9}$. Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base $e$, $\ln\left(e^{-2t}\right) = -2t$. We can also use the Power Rule to write $\ln\left(\frac{1}{9}\right) = -\ln(9)$. Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?) The calculator confirms the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$ intersect at $x = \ln(3) \approx 1.099$.




$y = f(x) = 9 \cdot 3^x$ and
$\boldsymbol{y = g(x) = 7^{2x}}$

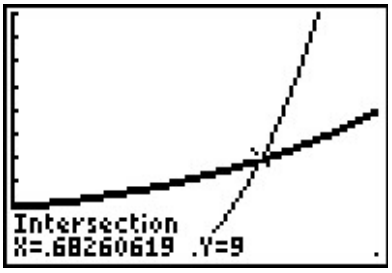



$y = f(x) = 75$ and
$\boldsymbol{y = g(x) = \frac{100}{1+3e^{-2x}}}$

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$. Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs. If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'. Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$. Since $5^x = -2$ has no real solution, (Why not?) we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln\left(5^x\right) = \ln(3)$ so that $x\ln(5) = \ln(3)$

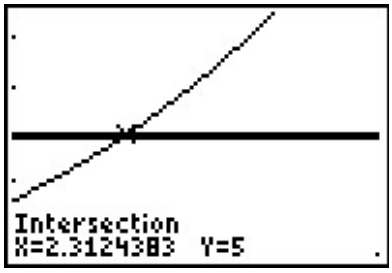
or $x = \frac{\ln(3)}{\ln(5)}$. When we graph $f(x) = 25^x$ and $g(x) = 5^x + 6$, we see that they intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.6826$.

6. At first, it's unclear how to proceed with $\frac{e^x - e^{-x}}{2} = 5$, besides clearing the denominator to obtain $e^x - e^{-x} = 10$. Of course, if we rewrite $e^{-x} = \frac{1}{e^x}$, we see we have another denominator lurking in the problem: $e^x - \frac{1}{e^x} = 10$. Clearing this denominator gives us $e^{2x} - 1 = 10e^x$, and once again, we have an equation with three terms where the exponent on one term is exactly twice that of another - a 'quadratic in disguise.' If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$, we obtain by the quadratic formula $u = 5 \pm \sqrt{26}$. From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$, but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$. If we graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$, we see that the graphs intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$




$y = f(x) = 25^x$ and

$\boldsymbol{y = g(x) = 5^x + 6}$




$y = f(x) = \frac{e^x - e^{-x}}{2}$ and

$\boldsymbol{y = g(x) = 5}$

□

The authors would be remiss not to mention that Example 7.3.1 still holds great educational value. Much can be learned about logarithms and exponentials by verifying the solutions obtained in Example 7.3.1 analytically. For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and obtain

$$\begin{aligned}
2000 &\stackrel{?}{=} 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} \\
2000 &\stackrel{?}{=} 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} \\
2000 &\stackrel{?}{=} 1000 \cdot 3^{\log_3(2)} && \text{Change of Base} \\
2000 &\stackrel{?}{=} 1000 \cdot 2 && \text{Inverse Property} \\
2000 &\stackrel{\checkmark}{=} 2000
\end{aligned}$$

The other solutions can be verified by using a combination of log and inverse properties. Some fall out quite quickly, while others are more involved. We leave them to the reader.

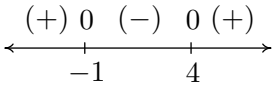
Since exponential functions are continuous on their domains, the Intermediate Value Theorem 4.1 applies. As with the algebraic functions in Section **??**, this allows us to solve inequalities using sign diagrams as demonstrated below.

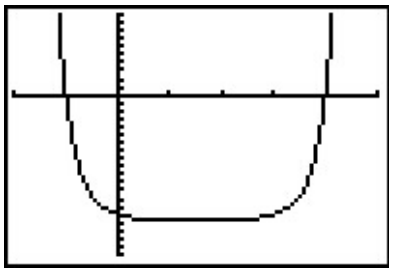
EXAMPLE 7.3.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $xe^{2x} < 4x$

SOLUTION.

1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$. The domain of $r$ is all real numbers, so in order to construct our sign diagram, we seed to find the zeros of $r$. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by the one-to-one property of exponential functions, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$, is on or above the $x$-axis.



$y = r(x) = 2^{x^2-3x} - 16$

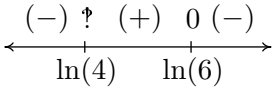
2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

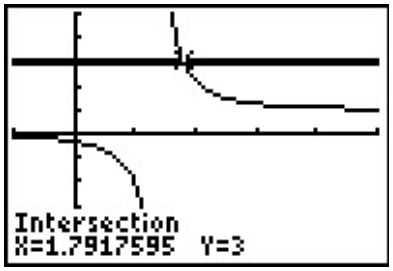
$$\begin{aligned}
\frac{e^x}{e^x - 4} &\leq 3 \\
\frac{e^x}{e^x - 4} - 3 &\leq 0 \\
\frac{e^x}{e^x - 4} - \frac{3\,(e^x - 4)}{e^x - 4} &\leq 0 \quad \text{Common denomintors.} \\
\frac{12 - 2e^x}{e^x - 4} &\leq 0
\end{aligned}$$

We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that $r$ is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of $r$ is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of $r$, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. Solving for $e^x$, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. While the prospect of determining the sign of $r(\ln(3))$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r(\ln(3)) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

We determine the signs of $r(\ln(5))$ and $r(\ln(7))$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$. Using the calculator, we see the graph of $f(x) = \frac{e^x}{e^x-4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x = \ln(6) \approx 1.792$.




$$y = f(x) = \frac{e^x}{e^x-4}$$

$$\boldsymbol{y = g(x) = 3}$$

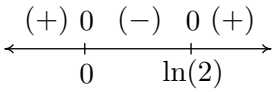
3. As before, we start solving $xe^{2x} < 4x$ by getting 0 on one side of the inequality, $xe^{2x} - 4x < 0$. We set $r(x) = xe^{2x} - 4x$ and since there are no denominators, even-indexed radicals, or logs, the domain of $r$ is all real numbers. Setting $r(x) = 0$ produces $xe^{2x} - 4x = 0$. With $x$ both in and out of the exponent, this could cause some difficulty. However, before panic sets in, we factor out the $x$ to obtain $x\left(e^{2x} - 4\right) = 0$ which gives $x = 0$ or $e^{2x} - 4 = 0$. To solve the latter, we isolate the exponential and take logs to get $2x = \ln(4)$, or $x = \frac{\ln(4)}{2} = \ln(2)$. (Can you explain the last equality using properties of logs?) As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we have $r(\ln\left(\frac{1}{2}\right)) = \ln\left(\frac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right)$. Applying the Power Rule to the log

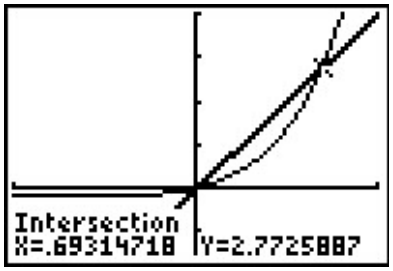
[4] This is because the base of $\ln(x)$ is $e > 1$. If the base $b$ were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

[5] We could, of course, use the calculator, but what fun would that be?

[6] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

in the exponent, we obtain $\ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right)$. Using the inverse properties of logs, this reduces to $\frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right)$. Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$. Continuing in this manner, we find $r(x) < 0$ on $(0, \ln(2))$. The calculator confirms that the graph of $f(x) = xe^{2x}$ is below the graph of $g(x) = 4x$ on this intervals.[7]




$y = f(x) = xe^{2x}$ and

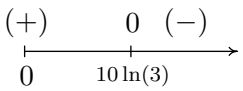
$\boldsymbol{y = g(x) = 4x}$

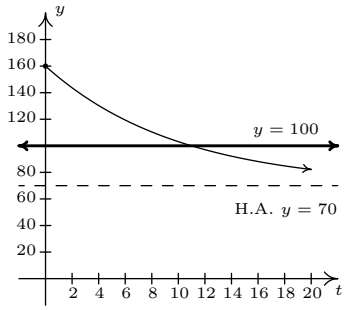
□

EXAMPLE 7.3.3. Recall from Example 7.1.2 that the temperature of coffee $T$ (in degrees Fahrenheit) $t$ minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

SOLUTION. We need to find when $T(t) > 100$, or in other words, we need to solve the inequality $70 + 90e^{-0.1t} > 100$. Getting 0 on one side of the inequality, we have $90e^{-0.1t} - 30 > 0$, and we set $r(t) = 90e^{-0.1t} - 30$. The domain of $r$ is artificially restricted due to the context of the problem to $[0, \infty)$, so we proceed to find the zeros of $r$. Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which, after a quick application of the Power Rule leaves us with $t = 10\ln(3)$. If we wish to avoid using the calculator to choose test values, we note that since $1 < 3$, $0 = \ln(1) < \ln(3)$ so that $10\ln(3) > 0$. So we choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $10\ln(3) < 10\ln(4)$, so the latter is our choice of a test value for the interval $(10\ln(3), \infty)$. Our sign diagram is below, and next to it is our graph of $t = T(t)$ from Example 7.1.2 with the horizontal line $y = 100$.

[7] Note: $\ln(2) \approx 0.693$.




$y = T(t)$

In order to interpret what this means in the context of the real world, we need a reasonable approximation of the number $10\ln(3) \approx 10.986$. This means it takes approximately 11 minutes for the coffee to cool to 100°F. Until then, the coffee is warmer than that.[8] □

We close this section by finding the inverse of a function which is a composition of a rational function with an exponential function.

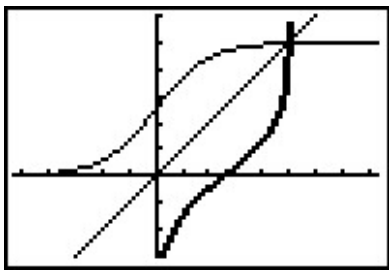
Example 7.3.4. The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution. We start by writing $y = f(x)$, and interchange the roles of $x$ and $y$. To solve for $y$, we first clear denominators and then isolate the exponential function.

$$\begin{aligned}
y &= \frac{5e^x}{e^x + 1} \\
x &= \frac{5e^y}{e^y + 1} \qquad \text{Switch } x \text{ and } y \\
x\left(e^y + 1\right) &= 5e^y \\
xe^y + x &= 5e^y \\
x &= 5e^y - xe^y \\
x &= e^y(5 - x) \\
e^y &= \frac{x}{5 - x} \\
\ln\left(e^y\right) &= \ln\left(\frac{x}{5 - x}\right) \\
y &= \ln\left(\frac{x}{5 - x}\right)
\end{aligned}$$

[8] Critics may point out that since we needed to use the calculator to interpret our answer anyway, why not use it earlier to simplify the computations? It is a fair question which we answer unfairly: it's our book.

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all $x$ in the domain of $f$ and that $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the domain of $f^{-1}$. We leave this to the reader. To verify our solution graphically, we graph $y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$ on the same set of axes and observe the symmetry about the line $y = x$. Note the domain of $f$ is the range of $g$ and vice-versa.



$y = f(x) = \frac{5e^x}{e^x+1}$ and $\boldsymbol{y = g(x) = \ln\left(\frac{x}{5-x}\right)}$

□

### 7.3.1 Exercises

1. Solve the following equations analytically.

   (a) $3^{(x-1)} = 27$
   (b) $3^{(x-1)} = 29$
   (c) $3^{(x-1)} = 2^x$
   (d) $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
   (e) $8^x = \frac{1}{128}$
   (f) $3^{7x} = 81^{4-2x}$
   (g) $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
   (h) $7e^{2x} = 28e^{-6x}$
   (i) $7^{3+7x} = 3^{4-2x}$
   (j) $\left(1 + \frac{0.06}{12}\right)^{12t} = 3$
   (k) $e^{-5730k} = \frac{1}{2}$
   (l) $2^{(x^3-x)} = 1$
   (m) $e^{2x} = 2e^x$
   (n) $70 + 90e^{-0.1t} = 75$
   (o) $\frac{150}{1+29e^{-0.8t}} = 75$
   (p) $25\left(\frac{4}{5}\right)^x = 10$

2. Solve the following inequalities analytically.

   (a) $e^x > 53$
   (b) $1000\left(1 + \frac{0.06}{12}\right)^{12t} \geq 3000$
   (c) $2^{(x^3-x)} < 1$
   (d) $25\left(\frac{4}{5}\right)^x \geq 10$
   (e) $\frac{150}{1+29e^{-0.8t}} \leq 130$
   (f) $70 + 90e^{-0.1t} \leq 75$

3. Use your calculator to help you solve the following equations and inequalities.

   (a) $e^x < x^3 - x$
   (b) $2^x = x^2$
   (c) $e^{\sqrt{x}} = x + 1$
   (d) $e^{-x} - xe^{-x} \geq 0$
   (e) $3^{(x-1)} < 2^x$
   (f) $e^x = \ln(x) + 5$

4. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram.

5. Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both $f$ and $f^{-1}$.

6. In Example 7.3.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5 - x}\right)$ but we left a few loose ends for you to tie up.

   (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all $x$ in the domain of $f$ and that $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the domain of $f^{-1}$.
   (b) Find the range of $f$ by finding the domain of $f^{-1}$.
   (c) Let $g(x) = \dfrac{5x}{x + 1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. [9]

7. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers $n$. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

[9] We know this is true in general by Exercise 8 in Section 6.2, but it's nice to see a specific example of the property.

### 7.3.2 ANSWERS

1. (a) $x = 4$
   (b) $x = \dfrac{\ln(29) + \ln(3)}{\ln(3)}$
   (c) $x = \dfrac{\ln(3)}{\ln(3) - \ln(2)}$
   (d) $x = \dfrac{\ln(3) + 5\ln\left(\frac{1}{2}\right)}{\ln(3) - \ln\left(\frac{1}{2}\right)}$
   (e) $x = -\frac{7}{3}$
   (f) $x = \frac{16}{15}$
   (g) $x = -\frac{2}{11}$
   (h) $x = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
   (i) $x = \dfrac{4\ln(3) - 3\ln(7)}{7\ln(7) + 2\ln(3)}$
   (j) $t = \dfrac{\ln(3)}{12\ln(1.005)}$
   (k) $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730}$
   (l) $x = -1,\ 0,\ 1$
   (m) $x = \ln(2)$
   (n) $t = 10\ln(18)$
   (o) $t = \dfrac{\ln\left(\frac{1}{29}\right)}{-0.8}$
   (p) $x = \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}$

2. (a) $(\ln(53), \infty)$
   (b) $\left[\dfrac{\ln(3)}{12\ln(1.005)}, \infty\right)$
   (c) $(-\infty, -1) \cup (0, 1)$
   (d) $\left(-\infty, \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right]$
   (e) $\left(-\infty, \dfrac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, -\frac{5}{4}\ln\left(\frac{2}{377}\right)\right]$
   (f) $\left[\dfrac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$

3. (a) $(2.3217, 4.3717)$
   (b) $x \approx -0.76666,\ x = 2,\ x = 4$
   (c) $x = 0$
   (d) $(-\infty, 1]$
   (e) $(-\infty, 2.7095)$
   (f) $x \approx 0.01866,\ x \approx 1.7115$

4. $x > \frac{1}{3}(\ln(6) + 1)$

5. $f^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$. Both $f$ and $f^{-1}$ have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

## 7.4 Logarithmic Equations and Inequalities

In Section 7.3 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from. For example, suppose we wish to solve $\log_2(x) = \log_2(5)$. Theorem 7.4 tells us that the *only* solution to this equation is $x = 5$. Now suppose we wish to solve $\log_2(x) = 3$. If we want to use Theorem 7.4, we need to rewrite 3 as a logarithm base 2. We can use Theorem 7.3 to do just that: $3 = \log_2\left(2^3\right) = \log_2(8)$. Our equation then becomes $\log_2(x) = \log_2(8)$ so that $x = 8$. However, we could have arrived at the same answer, in fewer steps, by using Theorem 7.3 to rewrite the equation $\log_2(x) = 3$ as $2^3 = x$, or $x = 8$. We summarize the two common ways to solve log equations below.

**Steps for Solving an Equation involving Logarithmic Fuctions**

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate the arguments of the log functions.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 7.4.1. Solve the following equations. Check your solutions graphically using a calculator.

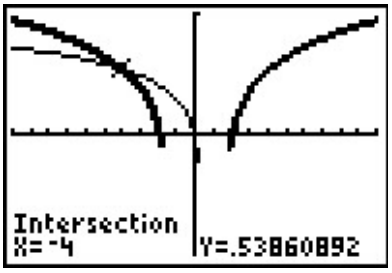
1. $\log_{117}(1 - 3x) = \log_{117}\left(x^2 - 3\right)$
2. $2 - \ln(x - 3) = 1$
3. $\log_6(x + 4) + \log_6(3 - x) = 1$
4. $\log_7(1 - 2x) = 1 - \log_7(3 - x)$
5. $\log_2(x + 3) = \log_2(6 - x) + 3$
6. $1 + 2\log_4(x + 1) = 2\log_2(x)$

Solution.

1. Since we have the same base on both sides of the equation $\log_{117}(1 - 3x) = \log_{117}\left(x^2 - 3\right)$, we equate what's inside the logs to get $1 - 3x = x^2 - 3$. Solving $x^2 + 3x - 4 = 0$ gives $x = -4$ and $x = 1$. To check these answers using the calculator, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$ and we see they intersect only at $x = -4$. To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.

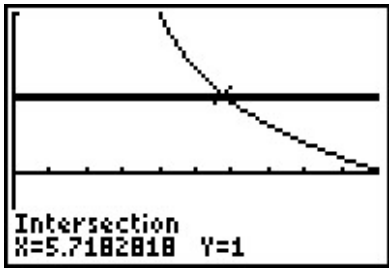
[1] They do, however, represent the same **family** of complex numbers. We stop ourselves at this point and refer the reader to a good course in Complex Variables.

2. Our first objective in solving $2-\ln(x-3)=1$ is to isolate the logarithm. We get $\ln(x-3)=1$, which, as an exponential equation, is $e^1 = x-3$. We get our solution $x = e+3$. On the calculator, we see the graph of $f(x) = 2-\ln(x-3)$ intersects the graph of $g(x) = 1$ at $x = e+3 \approx 5.718$.




$y = f(x) = \log_{117}(1-3x)$ and

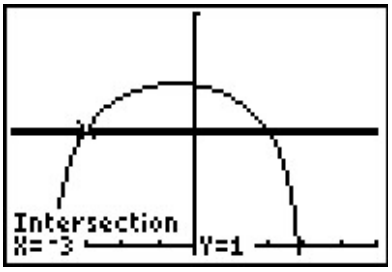
$\boldsymbol{y = g(x) = \log_{117}\left(x^2-3\right)}$

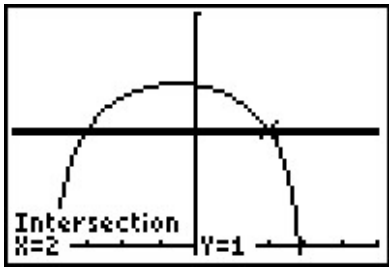



$y = f(x) = 2-\ln(x-3)$ and

$\boldsymbol{y = g(x) = 1}$

3. We can start solving $\log_6(x+4)+\log_6(3-x)=1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6[(x+4)(3-x)]=1$. Rewriting this as an exponential equation, we get $6^1 = (x+4)(3-x)$. This reduces to $x^2+x-6=0$, which gives $x=-3$ and $x=2$. Graphing $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$, we see they intersect twice, at $x=-3$ and $x=2$.



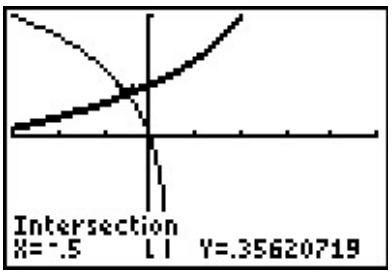




$y = f(x) = \log_6(x+4)+\log_6(3-x)$ and $\boldsymbol{y = g(x) = 1}$

4. Taking a cue from the previous problem, we begin solving $\log_7(1-2x) = 1-\log_7(3-x)$ by first collecting the logarithms on the same side, $\log_7(1-2x)+\log_7(3-x)=1$, and then using the Product Rule to get $\log_7[(1-2x)(3-x)]=1$. Rewriting this as an exponential equation gives $7^1 = (1-2x)(3-x)$ which gives the quadratic equation $2x^2-7x-4=0$. Solving, we find $x=-\frac{1}{2}$ and $x=4$. Graphing, we find $y = f(x) = \frac{\ln(1-2x)}{\ln(7)}$ and $y = g(x) = 1-\frac{\ln(3-x)}{\ln(7)}$ intersect only at $x=-\frac{1}{2}$. Checking $x=4$ in the original equation produces $\log_7(-7) = 1-\log_7(-1)$, which is a clear domain violation.

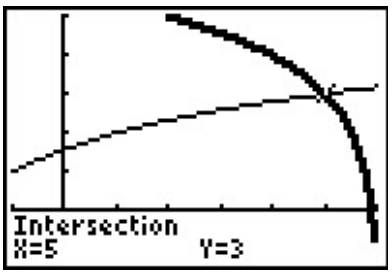
5. Starting with $\log_2(x+3) = \log_2(6-x)+3$, we gather the logarithms to one side and get $\log_2(x+3)-\log_2(6-x)=3$, and then use the Quotient Rule to obtain $\log_2\left(\frac{x+3}{6-x}\right) = 3$.

Rewriting this as an exponential equation gives $2^3 = \frac{x+3}{6-x}$. This reduces to the linear equation $8(6-x) = x+3$, which gives us $x = 5$. When we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$, we find they intersect at $x = 5$.




$y = f(x) = \log_7(1-2x)$ and

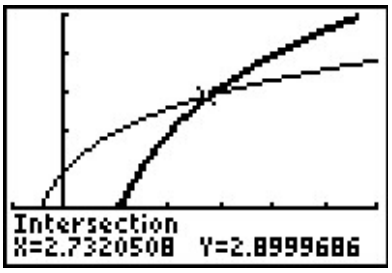
$\boldsymbol{y = g(x) = 1 - \log_7(3-x)}$




$y = f(x) = \log_2(x+3)$ and

$\boldsymbol{y = g(x) = \log_2(6-x) + 3}$

6. Starting with $1 + 2\log_4(x+1) = 2\log_2(x)$, we gather the logs to one side to get the equation $1 = 2\log_2(x) - 2\log_4(x+1)$. Before we can combine the logarithms, however, we need a common base. Since 4 is a power of 2, we use change of base to convert $\log_4(x+1) = \frac{\log_2(x+1)}{\log_2(4)} = \frac{1}{2}\log_2(x+1)$. Hence, our original equation becomes $1 = 2\log_2(x) - 2\left(\frac{1}{2}\log_2(x+1)\right)$ or $1 = 2\log_2(x) - \log_2(x+1)$. Using the Power and Quotient Rules, we obtain $1 = \log_2\left(\frac{x^2}{x+1}\right)$. Rewriting this in exponential form, we get $\frac{x^2}{x+1} = 2$ or $x^2 - 2x - 2 = 0$. Using the quadratic formula, we get $x = 1 \pm \sqrt{3}$. Graphing $f(x) = 1 + \frac{2\ln(x+1)}{\ln(4)}$ and $g(x) = \frac{2\ln(x)}{\ln(2)}$, we see the graphs intersect only at $x = 1 + \sqrt{3} \approx 2.732$. The solution $x = 1 - \sqrt{3} < 0$, which means if substituted into the original equation, the term $2\log_2\left(1 - \sqrt{3}\right)$ is undefined.




$y = f(x) = 1 + 2\log_4(x+1)$ and $\boldsymbol{y = g(x) = 2\log_2(x)}$

□

If nothing else, Example 7.4.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically,

[2]Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

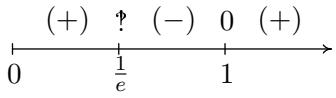
extraneous solutions are easy to spot - any supposed solution which causes a negative number inside a logarithm needs to be discarded. As with the equations in Example 7.3.1, much can be learned from checking all of the answers in Example 7.4.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

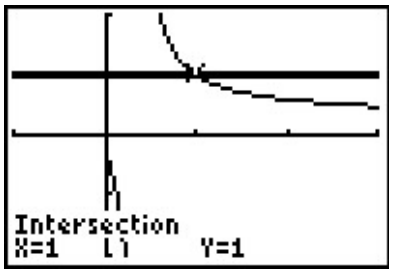
EXAMPLE 7.4.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1} \leq 1$ 2. $(\log_2(x))^2 < 2\log_2(x) + 3$ 3. $x\log(x+1) \geq x$

SOLUTION.

1. We start solving $\frac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1} - 1 \leq 0$. Getting a common denominator yields $\frac{1}{\ln(x)+1} - \frac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\frac{\ln(x)}{\ln(x)+1} \geq 0$. We define $r(x) = \frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of $r$. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x) + 1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of $r$ is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$. To find the zeros of $r$, we set $r(x) = \frac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$. In order to determine test values for $r$ without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of $e$. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$. To determine the sign of $r\left(\frac{1}{e^2}\right)$, we use the fact that $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$, and find $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2$, which is $(+)$. The rest of the test values are determined similarly. From our sign diagram, we find the solution to be $\left(0, \frac{1}{e}\right) \cup [1, \infty)$. Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see the the graph of $f$ is below the graph of $g$ on the solution intervals, and that the graphs intersect at $x = 1$.

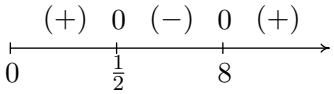


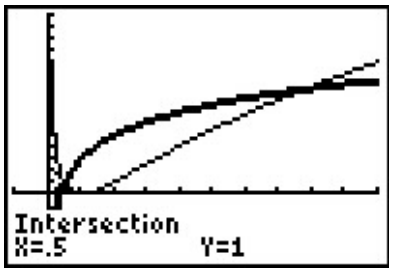




$y = f(x) = \frac{1}{\ln(x)+1}$ and $\boldsymbol{y = g(x) = 1}$

2. Moving all of the nonzero terms of $(\log_2(x))^2 < 2\log_2(x) + 3$ to one side of the inequality, we have $(\log_2(x))^2 - 2\log_2(x) - 3 < 0$. Defining $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$, we get the domain of $r$ is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of $r$, we

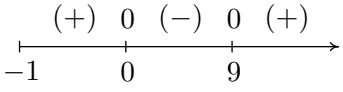
set $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3 = 0$ which results in a 'quadratic in disguise.' We set $u = \log_2(x)$ so our equation becomes $u^2 - 2u - 3 = 0$ which gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, which gives us $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which yields $x = 2^3 = 8$. We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$, and from our sign diagram, we see $r(x) < 0$ on $\left(\frac{1}{2}, 8\right)$. Geometrically, we see the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on the solution interval.

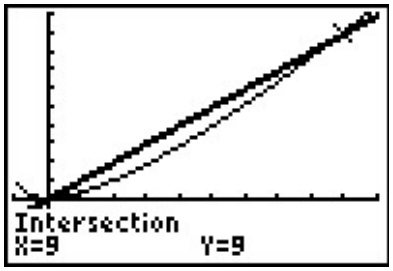


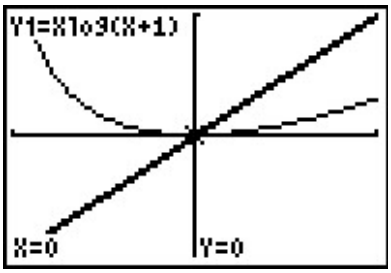

$y = f(x) = (\log_2(x))^2$ and $\boldsymbol{y = g(x) = 2\log_2(x) + 3}$

3. We begin to solve $x\log(x+1) \geq x$ by subtracting $x$ from both sides to get $x\log(x+1) - x \geq 0$. We define $r(x) = x\log(x+1) - x$ and due to the presence of the logarithm, we require $x+1 > 0$, or $x > -1$. To find the zeros of $r$, we set $r(x) = x\log(x+1) - x = 0$. Factoring, we get $x(\log(x+1) - 1) = 0$, which gives $x = 0$ or $\log(x+1) - 1 = 0$. The latter gives $\log(x+1) = 1$, or $x + 1 = 10^1$, which admits $x = 9$. We select test values $x$ so that $x + 1$ is a power of 10, and we obtain $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$. Our sign diagram gives the solution to be $(-1, 0] \cup [9, \infty)$. The calculator indicates the graph of $y = f(x) = x\log(x+1)$ is above $y = g(x) = x$ on the solution intervals, and the graphs intersect at $x = 0$ and $x = 9$.

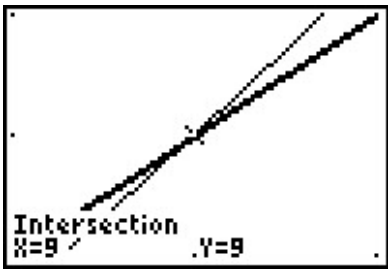



$y = f(x) = x\log(x+1)$ and $\boldsymbol{y = g(x) = x}$




Near $x = 0$



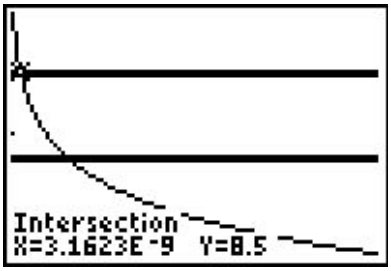

Near $x = 9$

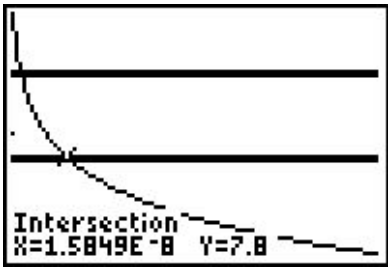
□

Our next example revisits the concept of pH as first introduced in the exercises in Section 7.1.

EXAMPLE 7.4.3. In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration.

SOLUTION. Recall from Exercise 6c in Section 7.1 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-7.8 \geq \log[\text{H}^+] \geq -8.5$. To solve this compound inequality we solve $-7.8 \geq \log[\text{H}^+]$ and $\log[\text{H}^+] \geq -8.5$ and take the intersection of the solution sets.[3] The former inequality yields $0 < [\text{H}^+] \leq 10^{-7.8}$ and the latter yields $[\text{H}^+] \geq 10^{-8.5}$. Taking the intersection gives us our final answer $10^{-8.5} \leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) After carefully adjusting the viewing window on the graphing calculator we see that the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.16 \times 10^{-9}, 1.58 \times 10^{-8}]$.




The graphs of $y = f(x) = -\log(x)$, $\boldsymbol{y = 7.8}$ and $\boldsymbol{y = 8.5}$

□

We close this section by finding an inverse of a one-to-one function which involves logarithms.

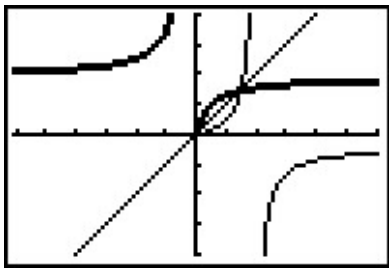
[3] Refer to page 202 for a discussion of what this means.

Example 7.4.4. The function $f(x) = \dfrac{\log(x)}{1-\log(x)}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution. We first write $y = f(x)$ then interchange the $x$ and $y$ and solve for $y$.

$$\begin{aligned}
y &= f(x) \\
y &= \frac{\log(x)}{1-\log(x)} \\
x &= \frac{\log(y)}{1-\log(y)} && \text{Interchange } x \text{ and } y. \\
x\,(1-\log(y)) &= \log(y) \\
x - x\log(y) &= \log(y) \\
x &= x\log(y) + \log(y) \\
x &= (x+1)\log(y) \\
\frac{x}{x+1} &= \log(y) \\
y &= 10^{\frac{x}{x+1}} && \text{Rewrite as an exponential equation.}
\end{aligned}$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing $f$ and $f^{-1}$ on the same viewing window yields



$$y = f(x) = \frac{\log(x)}{1 - \log(x)} \text{ and } \boldsymbol{y = g(x) = 10^{\frac{x}{x+1}}}$$

□

### 7.4.1 Exercises

1. Solve the following equations analytically.

   (a) $\log_{\frac{1}{2}} x = -3$
   (b) $\ln(x^2) = (\ln(x))^2$
   (c) $\log_3(x-4) + \log_3(x+4) = 2$
   (d) $\log_5(2x+1) + \log_5(x+2) = 1$
   (e) $\log_2(x^3) = \log_2(x)$
   (f) $\log_{169}(3x+7) - \log_{169}(5x-9) = \frac{1}{2}$
   (g) $\log\left(\frac{x}{10^{-3}}\right) = 4.7$
   (h) $-\log(x) = 5.4$
   (i) $10\log\left(\frac{x}{10^{-12}}\right) = 150$
   (j) $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$
   (k) $\log_{125}\left(\frac{3x-2}{2x+3}\right) = \frac{1}{3}$
   (l) $\ln(x+1) - \ln(x) = 3$
   (m) $\ln(\ln(x)) = 3$
   (n) $2\log_7(x) = \log_7(2) + \log_7(x+12)$
   (o) $\log(x) - \log(2) = \log(x+8) - \log(x+2)$

2. Solve the following inequalities analytically.

   (a) $x\ln(x) - x > 0$
   (b) $5.6 \leq \log\left(\frac{x}{10^{-3}}\right) \leq 7.1$
   (c) $10\log\left(\frac{x}{10^{-12}}\right) \geq 90$
   (d) $2.3 < -\log(x) < 5.4$
   (e) $\frac{1-\ln(x)}{x^2} < 0$
   (f) $\ln(x^2) \leq (\ln(x))^2$

3. Use your calculator to help you solve the following equations and inequalities.

   (a) $\ln(x) = e^{-x}$
   (b) $\ln(x^2+1) \geq 5$
   (c) $\ln(x) = \sqrt[4]{x}$
   (d) $\ln(-2x^3 - x^2 + 13x - 6) < 0$

4. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x+1) < 3$ without a sign diagram. Also, compare this exercise to question 4 in Section 7.3.

5. Solve $\ln(3-y) - \ln(y) = 2x + \ln(5)$ for $y$.

6. In Example 7.4.4 we found the inverse of $f(x) = \dfrac{\log(x)}{1-\log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

   (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all $x$ in the domain of $f$ and that $\left(f \circ f^{-1}\right)(x) = x$ for all $x$ in the domain of $f^{-1}$.
   (b) Find the range of $f$ by finding the domain of $f^{-1}$.
   (c) Let $g(x) = \dfrac{x}{1-x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise 8 in Section 6.2, but it's nice to see a specific example of the property.)

7. Let $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

8. Explain the equation in Exercise 1g and the inequality in Exercise 2b above in terms of the Richter scale for earthquake magnitude. (See Exercise 6a in Section 7.1.)

9. Explain the equation in Exercise 1i and the inequality in Exercise 2c above in terms of sound intensity level as measured in decibels. (See Exercise 6b in Section 7.1.)

10. Explain the equation in Exercise 1h and the inequality in Exercise 2d above in terms of the pH of a solution. (See Exercise 6c in Section 7.1.)

11. With the help of your classmates, solve the inequality $\sqrt[n]{x} > \ln(x)$ for a variety of natural numbers $n$. What might you conjecture about the "speed" at which $f(x) = \ln(x)$ grows versus any principal $n^{\text{th}}$ root function?

### 7.4.2 Answers

1. (a) $x = 8$
   (b) $x = 1$, $x = e^2$
   (c) $x = 5$
   (d) $x = \frac{1}{2}$
   (e) $x = 1$
   (f) $x = 2$
   (g) $x = 10^{1.7}$
   (h) $x = 10^{-5.4}$
   (i) $x = 10^3$
   (j) $x = 81$
   (k) $x = -\frac{17}{7}$
   (l) $x = \frac{1}{e^3 - 1}$
   (m) $x = e^{e^3}$
   (n) $x = 6$
   (o) $x = 4$

2. (a) $(e, \infty)$
   (b) $\left[10^{2.6}, 10^{4.1}\right]$
   (c) $\left[10^{-3}, \infty\right)$
   (d) $\left(10^{-5.4}, 10^{-2.3}\right)$
   (e) $(e, \infty)$
   (f) $(0, 1] \cup [e^2, \infty)$

3. (a) $x \approx 1.3098$
   (b) $(-\infty, -12.1414) \cup (12.1414, \infty)$
   (c) $x \approx 4.177$, $x \approx 5503.665$
   (d) $(-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$

4. $-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$

5. $y = \dfrac{3}{5e^{2x} + 1}$

7. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. (The reason for this rewriting will be explained much later in the text.) The domain of $f^{-1}$ is $(-\infty, \infty)$ and its range is the same as the domain of $f$, namely $(-1, 1)$.

## 7.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

As we mentioned in Section 7.1, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, the calculator is often used to express our answers as decimal approximations.

### 7.5.1 APPLICATIONS OF EXPONENTIAL FUNCTIONS

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have \$100 to invest at your local bank and they are offering a whopping 5 % annual percentage interest rate. This means that after one year, the bank will pay **you** 5% of that \$100, or \$100(0.05) = \$5 in interest, so you now have \$105.[1] This is in accordance with the formula for **simple interest** which you have undoubtedly run across at some point in your mathematical upbringing.

EQUATION 7.1. **Simple Interest** The amount of interest $I$ accrued at an annual rate $r$ on an investment[a] $P$ after $t$ years is

$$I = Prt$$

The amount $A$ in the account after $t$ years is given by

$$A = P + I = P + Prt = P(1 + rt)$$

[a] Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[2] Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial \$100 yields \$100(0.05) $\left(\frac{1}{2}\right)$ = \$2.50 in interest. You take your \$102.50 off to the competitor and find out that those restrictions which *may* apply actually do apply to you, and you return to your bank which happily accepts your \$102.50 for the remaining six months of the year. To your surprise and delight, at the end of the year your statement reads \$105.06, not \$105 as you had expected.[3] Where did those extra six cents come from? For the first six months of the year, interest was earned on the original principal of \$100, but for the second six months, interest was earned on \$102.50, that is, you earned interest on your interest. This is the basic concept behind **compound interest**. In the previous discussion, we would say that the interest was compounded twice, or semiannually.[4] If more money can be earned by earning interest

[1] How generous of them!
[2] Some restrictions may apply.
[3] Actually, the final balance should be \$105.0625.
[4] Using this convention, simple interest after one year is the same as compounding the interest only once.

on interest already earned, a natural question to ask is what happens if the interest is compounded more often, say 4 times a year, which is every three months, or 'quarterly.' In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1 + rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the \$101.25 for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is \$103.79, and, at last, we obtain \$105.08. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound. In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal $P$ at an annual rate $r$ and compound the interest $n$ times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let $A_k$ denote the amount in the account after the $k^{\text{th}}$ compounding. Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use $A_1$ as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$. Since we compound the interest $n$ times per year, after $t$ years, we have $nt$ compoundings. We have just derived the general formula for compound interest below.

Equation 7.2. **Compounded Interest:** If an initial principal $P$ is invested at an annual rate $r$ and the interest is compounded $n$ times per year, the amount $A$ in the account after $t$ years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 7.2 becomes $A = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A = 100(1.0125)^{4t}$. This equation defines the amount $A$ as an exponential function of time $t$, $A(t)$. To check this against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 7.5.1. Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.

1. Express the amount $A$ in the account as a function of the term of the investment $t$ in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change[5] of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution.

[5] See Definition 3.3 in Section 3.1.

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (monthly) into Equation 7.2 yields $A = 2000\left(1 + \frac{0.07125}{12}\right)^{12t}$. Using function notation, we get $A(t) = 2000(1.0059375)^{12t}$.

2. Since $t$ represents the length of the investment, we substitute $t = 5$ into $A(t)$ to find $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately \$2852.92.

3. Our initial investment is \$2000, so to find the time it takes this to double, we need to find $t$ when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$. Taking natural logs as in Section 7.3, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. To find the average rate of change of $A$ from the end of the fourth year to the end of the fifth year, we compute $\frac{A(5)-A(4)}{5-4} \approx 195.63$. Similarly, the average rate of change of $A$ from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$. This means that the value of the investment is increasing at a rate of approximately \$195.63 per year between the end of the fourth and fifth years, while that rate jumps to \$1648.21 per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[6] □

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[7] Consider an investment of \$1 invested at 100% interest for 1 year compounded $n$ times a year. Equation 7.2 tells us that the amount of money in the account after 1 year is $A = \left(1 + \frac{1}{n}\right)^n$. Below is a table of values relating $n$ and $A$.

| $n$ | $A$ |
|---|---|
| 1 | 2 |
| 2 | 2.25 |
| 4 | $\approx 2.4414$ |
| 12 | $\approx 2.6130$ |
| 360 | $\approx 2.7145$ |
| 1000 | $\approx 2.7169$ |
| 10000 | $\approx 2.7181$ |
| 100000 | $\approx 2.7182$ |

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing. We are witnessing a mathematical

[6] In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

[7] Once you've had a semester of Calculus, you'll be able to fully appreciate this very lame pun.

'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest. With Calculus, we can show[8] that as $n \to \infty$, $A = \left(1 + \frac{1}{n}\right)^n \to e$, where $e$ is the natural base first presented in Section 7.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 7.8. If you invest \$1 at 100% interest compounded continuously, then you will have \$$e$ at the end of one year.

Using this definition of $e$ and a little Calculus, we can take Equation 7.2 and produce a formula for continuously compounded interest.

Equation 7.3. **Continuously Compounded Interest:** If an initial principal $P$ is invested at an annual rate $r$ and the interest is compounded continuously, the amount $A$ in the account after $t$ years is

$$A = Pe^{rt}$$

If we take the scenario of Example 7.5.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about \$24,035.28, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about \$24,213.18 - a difference of less than 1%.

Equations 7.2 and 7.3 both use exponential functions to describe the growth of an investment. Curiously enough, the same principles which govern compound interest are also used to model short term growth of populations. In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time.[9] In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Its solution is stated below.

[8] Or define, depending on your point of view.

[9] The average rate of change of a function over an interval was first introduced in Section 3.1. *Instantaneous* rates of change are the business of Calculus, as is mentioned on Page 167.

EQUATION 7.4. **Uninhibited Growth:** If a population increases according to The Law of Uninhibited Growth, the number of organisms $N$ at time $t$ is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read '$N$ nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$\text{(instantaneous rate of change of } N(t) \text{ at time } t) = k\, N(t)$$

It is worth taking some time to compare Equations 7.3 and 7.4. In Equation 7.3, we use $P$ to denote the initial investment; in Equation 7.4, we use $N_0$ to denote the initial population. In Equation 7.3, $r$ denotes the annual interest rate, and so it shouldn't be too surprising that the $k$ in Equation 7.4 corresponds to a growth rate as well. While Equations 7.3 and 7.4 look entirely different, they both represent the same mathematical concept.

EXAMPLE 7.5.2. In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, $N$, in thousands, after $t$ days.

SOLUTION. We begin with $N(t) = N_0 e^{kt}$. Since $N$ is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$. In order to complete the formula, we need to determine the growth rate $k$. We know that after one week, the number of cells has grown to five million. Since $t$ measures days and the units of $N$ are in thousands, this translates mathematically to $N(7) = 5000$. We get the equation $12e^{7k} = 5000$ which gives $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$. Hence, $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$. Of course, in practice, we would approximate $k$ to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells. □

Whereas Equations 7.3 and 7.4 model the growth of quantities, we can use equations like them to describe the decline of quantities. One example we've seen already is Example 7.1.1 in Section 7.1. There, the value of a car declined from its purchase price of $25,000 to nothing at all. Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays. This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 7.4 with the exception that the rate constant $k$ is negative.

Equation 7.5. **Radioactive Decay** The amount of a radioactive element $A$ at time $t$ is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$\text{(instantaneous rate of change of } A(t) \text{ at time } t) = k\, A(t)$$

Example 7.5.3. Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 7.5, and that the half-life[10] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, $A$, in grams, $t$ days later.

Solution. Since we start with 5 grams initially, Equation 7.5 gives $A(t) = 5e^{kt}$. Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Hence, $A(8) = 2.5$ which means $5e^{8k} = 2.5$. Solving, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily. Hence, $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$. □

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 7.1.2 of Section 7.1. In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where $t$ was measured in minutes. In this situation, we know the physical limit of the temperature of the coffee is room temperature,[11] and the differential equation which gives rise to our formula for $T(t)$ takes this into account. Whereas the radioactive decay model had a rate of decay at time $t$ directly proportional to the amount of the element which remained at time $t$, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time $t$ is directly proportional to how much of a temperature gap exists between the coffee at time $t$ and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly. Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

[10] The time it takes for half of the substance to decay.

[11] The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

EQUATION 7.6. **Newton's Law of Cooling (Warming):** The temperature $T$ of an object at time $t$ is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, $T_a$ is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } T(t) \text{ at time } t) = k\ (T(t) - T_a)$$

[a] That is, the temperature of the surroundings.

If we re-examine the situation in Example 7.1.2 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 7.6, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given. The rate constant $k = 0.1$ indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings. Note in Equation 7.6 that the constant $k$ is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if $T_0$ (the initial temperature of the object) is greater than $T_a$ (the ambient temperature) or vice-versa, as we see in our next example.

EXAMPLE 7.5.4. A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast $T$ as a function of its time in the oven, $t$, in hours.
2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

SOLUTION.

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming tells us $T(t) = 350 + (40 - 350)e^{-kt}$, or $T(t) = 350 - 310e^{-kt}$. To determine $k$, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is

$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. To determine when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. It takes roughly 3 hours and 15 minutes to cook the roast completely. □

If we had taken the time to graph $y = T(t)$ in Example 7.5.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion by applying a bit of 'number sense'. As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of $t$, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.[12] The function $T$ is sometimes called a **limited** growth model, since the function $T$ remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate. The **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Equation 7.7. **Logistic Growth:** If a population behaves according to the assumptions of logistic growth, the number of organisms $N$ at time $t$ is given by the equation

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, $L$ is the limiting population[a], $C$ is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)\,(L - N(t))$$

[a] That is, as $t \to \infty$, $N(t) \to L$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[13]

Example 7.5.5. The number of people $N$, in hundreds, at a local community college who have heard the rumor 'Carl is afraid of Virginia Woolf' can be modeled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

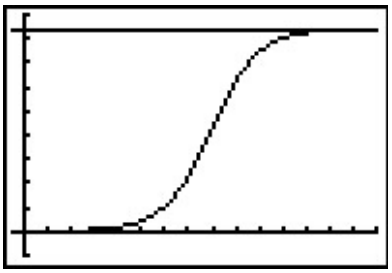
where $t \geq 0$ is the number of days after April 1, 2009.

[12] at which point it would be more toast than roast.
[13] Which can be just as damaging as diseases.

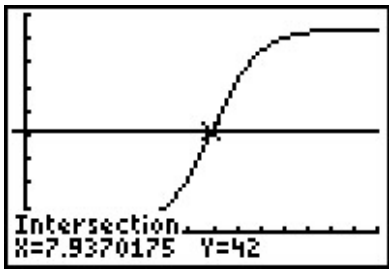
1. Find and interpret $N(0)$.
2. Find and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?
4. Check your answers to 2 and 3 using your calculator.

SOLUTION.

1. We find $N(0) = \frac{84}{1+2799e^0} = \frac{84}{2800} = \frac{3}{100}$. Since $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2009, we may conclude that on that day, 3 people have heard the rumor.[14]
2. We could simply note that $N(t)$ is written in the form of Equation 7.7, and identify $L = 84$. However, to see why the answer is 84, we proceed analytically. Since the domain of $N$ is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[15] as $t \to \infty$, have $1997e^{-t} \to 0^+$ and so $N(t) \approx \frac{84}{1+\text{very small } (+)} \approx 84$. Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.
3. To find how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$. It takes around 8 days until 4200 people have heard the rumor.
4. We graph $y = N(x)$ using the calculator and see that the line $y = 84$ is the horizontal asymptote of the graph, confirming our answer to part 2, and the graph intersects the line $y = 42$ at $x = \ln(2799) \approx 7.937$, which confirms our answer to part 3.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 84$



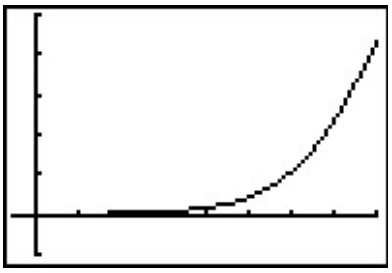

$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 42$

□

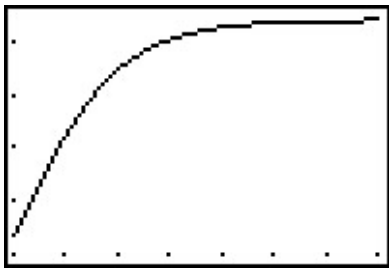
[14] Or, more likely, three people started the rumor. I'd wager Jeff, Jamie, and Jason started it. So much for telling your best friends something in confidence!

[15] See, for example, Example 7.1.2.

If we take the time to analyze the graph of $y = N(x)$ above, we can see graphically how logistic growth combines features of uninhibited and limited growth. The curve seems to rise steeply, then at some point, begins to level off. The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. At this point, even though the function is still increasing, the rate at which it does so begins to decline. It turns out the point of diminishing returns always occurs at half the limiting population. (In our case, when $y = 42$.) While these concepts are more precisely quantified using Calculus, below are two views of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth; the latter like limited growth.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $0 \leq x \leq 8$



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $8 \leq x \leq 16$

### 7.5.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 6a, 6b and 6c of Section 7.1, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 7.5.6. The information entropy $H$, in bits, of a randomly generated password consisting of $L$ characters is given by $H = L\log_2(N)$, where $N$ is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[16] password is comprised of letters and numbers only, find the associated information entropy.
2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution.

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

[16] That is, upper and lower case letters are treated as different characters.

2. We have $L = 7$ and $H = 50$ and we need to find $N$. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[17] □

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 6c in Section 7.1.

EXAMPLE 7.5.7. Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[18] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where $x$ is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

SOLUTION. We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr. □

[17] Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.

[18] Derived from the Henderson-Hasselbalch Equation. See Exercise 8 in Section 7.2. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

### 7.5.3 Exercises

1. On May, 31, 2009, the Annual Percentage Rate listed at my bank for regular savings accounts was 0.25% compounded monthly. Use Equation 7.2 to answer the following.

   (a) If $P = 2000$ what is $A(8)$?

   (b) Solve the equation $A(t) = 4000$ for $t$.

   (c) What principal $P$ should be invested so that the account balance is \$2000 is three years?

2. My bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

   (a) If $P = 2000$ what is $A(8)$?

   (b) Solve the equation $A(t) = 4000$ for $t$.

   (c) What principal $P$ should be invested so that the account balance is \$2000 is three years?

   (d) The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

3. Use Equation 7.2 to show that the time it takes for an investment to double in value does not depend on the principal $P$, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates $r$. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[19] in financial mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

4. Use Equation 7.5 to show that $k = -\dfrac{\ln(2)}{h}$ where $h$ is the half-life of the radioactive isotope.

5. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

   (a) Use Equation 7.5 to express the amount of Carbon-14 left from an initial $N$ milligrams as a function of time $t$ in years.

   (b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

   (c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

   (d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

[19] Awesome pun!

6. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 7.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time $t$ in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

7. In Example 7.1.1 in Section 7.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use the properties of logs and/or exponents to rewrite the model in the form $V(t) = 25e^{kt}$.

8. A pork roast was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F.[20] Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 7.6),

   (a) Express the temperature $T$ as a function of time $t$.

   (b) Find the time at which the roast would have dropped to 140°F had it not been carved and eaten.

[20] This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

### 7.5.4 Answers

1. (a) $A(8) = 2000\left(1 + \frac{0.0025}{12}\right)^{12 \cdot 8} \approx \$2040.40$
   (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0025}{12}\right)} \approx 277.29$ years
   (c) $P = \dfrac{2000}{\left(1 + \frac{0.0025}{12}\right)^{36}} \approx \$1985.06$

2. (a) $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12 \cdot 8} \approx \$2394.03$
   (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years
   (c) $P = \dfrac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$
   (d) $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

5. (a) $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$
   (b) $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains
   (c) $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old

6. $A(t) = 2.3e^{-0.0138629t}$

7. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$

8. (a) $T(t) = 75 + 105e^{-0.005005t}$
   (b) The roast would have cooled to 140°F in about 95 minutes.

# Chapter 8

# Systems of Equations

## 8.1 Systems of Linear Equations: Gaussian Elimination

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The $x$-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the $y$-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to $x$ and $y$ are of interest, we have what is known as a **system of equations**, usually written as

$$\begin{cases} y &= f(x) \\ y &= g(x) \end{cases}$$

The 'curly bracket' notation means we are to find all **pairs** of points $(x, y)$ which satisfy **both** equations. We begin our study of systems of equations by reviewing some basic notions from Intermediate Algebra.

Definition 8.1. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where $a_1$, $a_2$ and $c$ are real numbers and at least one of $a_1$ and $a_2$ is nonzero.

For reasons which will become clear later in the section, we are using subscripts in Definition 8.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$.[1] If $a_1$ and $a_2$ are both 0, then depending on $c$, we get either an

[1] Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. We leave it to the reader to explain why these do not satisfy Definition 8.1. From what we know from Sections 1.2 and 3.1, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example reviews some of the basic techniques first learned in Intermediate Algebra.

EXAMPLE 8.1.1. Solve the following systems of equations. Check your answer algebraically and graphically.

1. $\begin{cases} 2x - y = 1 \\ y = 3 \end{cases}$
2. $\begin{cases} 3x + 4y = -2 \\ -3x - y = 5 \end{cases}$
3. $\begin{cases} \frac{x}{3} - \frac{4y}{5} = \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} = \frac{1}{2} \end{cases}$
4. $\begin{cases} 2x - 4y = 6 \\ 3x - 6y = 9 \end{cases}$
5. $\begin{cases} 6x + 3y = 9 \\ 4x + 2y = 12 \end{cases}$
6. $\begin{cases} x - y = 0 \\ x + y = 2 \\ -2x + y = -2 \end{cases}$

SOLUTION.

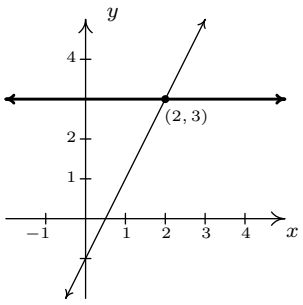
1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of $x$, we **substitute** this value for $y$ into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

2. To solve the second system, we use the **addition** method to **eliminate** the variable $x$. We take the two equations as given and 'add equals to equals' to obtain

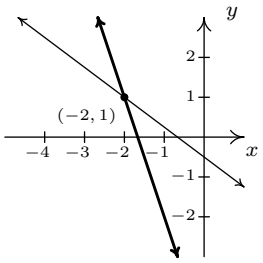
$$\begin{array}{rcl} 3x + 4y & = & -2 \\ + \quad (-3x - y & = & 5) \\ \hline 3y & = & 3 \end{array}$$

This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$

into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.




$2x - y = 1$
$\boldsymbol{y = 3}$




$3x + 4y = -2$
$\boldsymbol{-3x - y = 5}$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\begin{cases} 5x - 12y &= 21 \\ 4x + 6y &= 9 \end{cases}$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by $-5$, we will be in a position to eliminate the $x$ term

$$\begin{array}{rrcr} & 20x - 48y &=& 84 \\ + & (-20x - 30y &=& -45) \\ \hline & -78y &=& 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice − in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.

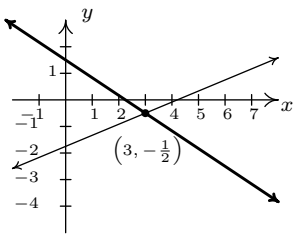
4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and the both sides of the second equation by $-2$, we are ready to eliminate the $x$

$$\begin{array}{rrcr} & 6x - 12y & = & 18 \\ + & (-6x + 12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

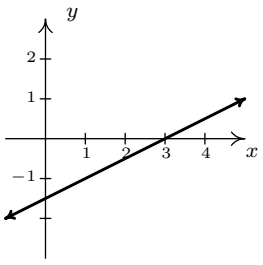
We eliminated not only the $x$, but the $y$ as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair $(x, y)$ satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$. One way to describe the solution set to this system is to use the roster method[2] and write $\{(x, y) : 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution**. Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of $x$, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding $y$-value of a solution. Since we have no restriction on $x$, it is called a **free variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t, \frac{1}{2}t - \frac{3}{2}\right) : -\infty < t < \infty\right\}$.[3] For specific values of $t$, we can generate solutions. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$; $t = 117$ gives us $(117, 57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always. We claim that for any real number $t$, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too. Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.

[2] See Section 1.2 for a review of this.

[3] Note that we could have just as easily chosen to solve $2x - 4y = 6$ for $x$ to obtain $x = 2y + 3$. Letting $y$ be the parameter $t$, we have that for any value of $t$, $x = 2t + 3$, which gives $\{(2t + 3, t) : -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.




$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$
$\mathbf{\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}}$




$2x - 4y = 6$
$\mathbf{3x - 6y = 9}$
(Same line.)

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by $-3$, we set the stage to eliminate $x$

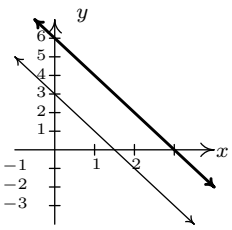
$$\begin{array}{rrcr} & 12x + 6y & = & 18 \\ + & (-12x - 6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

As in the previous example, both $x$ and $y$ dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair $(x, y)$ which satisfies both equations; in other words, the system has no solution. Graphically, we see that the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, and as such do not intersect.

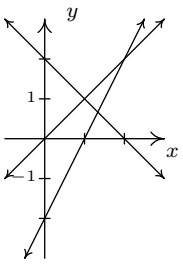
6. We can begin to solve our last system by adding the first two equations

$$\begin{array}{rrcr} & x - y & = & 0 \\ + & (x + y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x = 1$. Substituting this into the first equation gives $1 - y = 0$ so that $y = 1$. We seem to have determined a solution to our system, $(1, 1)$. While this checks in the first two equations, when we substitute $x = 1$ and $y = 1$ into the third equation, we get $-2(1)+(1) = -2$ which simplifies to the contradiction $-1 = -2$. Graphing the lines $x-y = 0$, $x + y = 2$, and $-2x + y = -2$, we see that the first two lines do, in fact, intersect at $(1, 1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.




$6x + 3y = 9$
$\mathbf{4x + 2y = 12}$




$y - x = 0$
$y + x = 2$
$-2x + y = -2$

□

A few remarks about Example 8.1.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[4] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 8.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

In order to move this section beyond a review of Intermediate Algebra, we now define what is meant by a linear equation in $n$ variables.

[4] In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5] The adjectives 'dependent' and 'independent' apply only to *consistent* systems - they describe the type of solutions.

[6] If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7] We need more than two variables to give an example of the latter.

[8] Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

Definition 8.2. A **linear equation in $n$ variables**, $x_1, x_2, \ldots, x_n$ is an equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ where $a_1, a_2, \ldots a_n$ and $c$ are real numbers and at least one of $a_1, a_2, \ldots, a_n$ is nonzero. Coupling more than one linear equation in $n$ variables results in a **system of linear equations in $n$ variables**.

Instead of using more familiar variables like $x$, $y$, and even $z$ and/or $w$ in Definition 8.2, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.) As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables $x_1$ and $x_2$ as the equation $3x - y = 4$ does between the variables $x$ and $y$. In addition, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$.

When solving a system, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations we've already employed. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[9]

Theorem 8.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a] That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

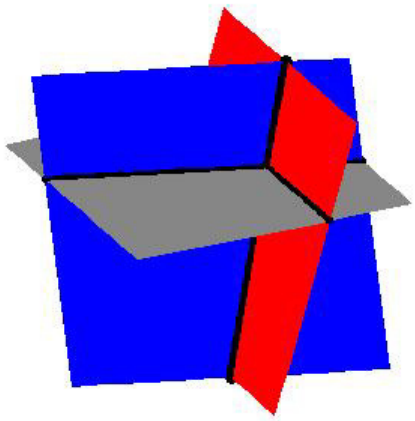
We have seen plenty of instances of the second and third moves in Theorem 8.1 when we solved the systems Example 8.1.1. The first move, while it obviously admits an equivalent system, seems silly. Our perception will change as we consider more equations and more variables in this, and later sections.

Consider the system of equations

$$\begin{cases} x - \frac{1}{3}y + \frac{1}{2}z &= 1 \\ y - \frac{1}{2}z &= 4 \\ z &= -1 \end{cases}$$

[9] That is, a system with the same solution set.

Clearly $z = -1$, and we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Finally, we substitute $y = \frac{7}{2}$ and $z = -1$ into the first equation to get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$, so that $x = \frac{8}{3}$. The reader can verify that these values of $x$, $y$ and $z$ satisfy all three original equations. It is tempting for us to write the solution to this system by extending the usual $(x, y)$ notation to $(x, y, z)$ and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[10] Moreover, just as equations involving the variables $x$ and $y$ describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables $x$, $y$, and $z$ describe objects called **surfaces** in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[11] so our intersection point is where all three of these lines meet.



Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

$$\begin{cases} x - \frac{1}{3}y + \frac{1}{2}z &= 1 \\ y - \frac{1}{2}z &= 4 \\ z &= -1 \end{cases}$$

[10] You were asked to think about this in Exercise 13 in Section 1.1.

[11] In fact, these lines are described by the parametric solutions to the systems formed by taking any two of these equations by themselves.

we note the reason it was so easy to solve is that the third equation is solved for $z$, the second equation involves only $y$ and $z$, and since the coefficient of $y$ is 1, it makes it easy to solve for $y$ using our known value for $z$. Lastly, the coefficient of $x$ in the first equation is 1 making it easy to substitute the known values of $y$ and $z$ and then solve for $x$. We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the **leading variable** of that equation.

Definition 8.3. A system of linear equations with variables $x_1$, $x_2$, $\ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.
2. The leading variable in each equation has coefficient 1.
3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.
4. Any equation without variables[a] cannot be placed above an equation with variables.

[a] necessarily an identity or contradiction

In our previous system, if make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[12] An example of a more complicated system in triangular form is

$$\begin{cases} x_1 - 4x_3 + x_4 - x_6 &= 6 \\ x_2 + 2x_3 &= 1 \\ x_4 + 3x_5 - x_6 &= 8 \\ x_5 + 9x_6 &= 10 \end{cases}$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 8.1.

Example 8.1.2. Use Theorem 8.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

[12] If letters are used instead of subscripted variables, Definition 8.3 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

1. $\begin{cases} 3x - y + z &= 3 \\ 2x - 4y + 3z &= 16 \\ x - y + z &= 5 \end{cases}$ 2. $\begin{cases} 2x + 3y - z &= 1 \\ 10x - z &= 2 \\ 4x - 9y + 2z &= 5 \end{cases}$ 3. $\begin{cases} 3x_1 + x_2 + x_4 &= 6 \\ 2x_1 + x_2 - x_3 &= 4 \\ x_2 - 3x_3 - 2x_4 &= 0 \end{cases}$

SOLUTION.

1. For definitiveness, we label the topmost equation in the system $E1$, the equation beneath that $E2$, and so forth. We now attempt to put the system in triangular form using an algorithm known as **Gaussian Elimination**. What this means is that, starting with $x$, we transform the system so that conditions 2 and 3 in Definition 8.3 are satisfied. Then we move on to the next variable, in this case $y$, and repeat. Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an $x$ in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 8.1 and interchange $E1$ and $E3$.

$$\begin{cases} (E1) & 3x - y + z &= 3 \\ (E2) & 2x - 4y + 3z &= 16 \\ (E3) & x - y + z &= 5 \end{cases} \xrightarrow{\text{Switch } E1 \text{ and } E3} \begin{cases} (E1) & x - y + z &= 5 \\ (E2) & 2x - 4y + 3z &= 16 \\ (E3) & 3x - y + z &= 3 \end{cases}$$

To satisfy Definition 8.3, we need to eliminate the $x$'s from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the $x$ from $E2$, we need to multiply $E1$ by $-2$ then add; to eliminate the $x$ from $E3$, we need to multiply $E1$ by $-3$ then add. Applying the third move listed in Theorem 8.1 twice, we get

$$\begin{cases} (E1) & x - y + z &= 5 \\ (E2) & 2x - 4y + 3z &= 16 \\ (E3) & 3x - y + z &= 3 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -3E1 + E3]{\text{Replace } E2 \text{ with } -2E1 + E2} \begin{cases} (E1) & x - y + z &= 5 \\ (E2) & -2y + z &= 6 \\ (E3) & 2y - 2z &= -12 \end{cases}$$

Now we enforce the conditions stated in Definition 8.3 for the variable $y$. To that end we need to get the coefficient of $y$ in $E2$ equal to 1. We apply the second move listed in Theorem 8.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\begin{cases} (E1) & x - y + z &= 5 \\ (E2) & -2y + z &= 6 \\ (E3) & 2y - 2z &= -12 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \begin{cases} (E1) & x - y + z &= 5 \\ (E2) & y - \frac{1}{2}z &= -3 \\ (E3) & 2y - 2z &= -12 \end{cases}$$

To eliminate the $y$ in $E3$, we add $-2E2$ to it.

$$\begin{cases} (E1) & x-y+z &=& 5 \\ (E2) & y-\frac{1}{2}z &=& -3 \\ (E3) & 2y-2z &=& -12 \end{cases} \xrightarrow{\text{Replace } E3 \text{ with } -2E2+E3} \begin{cases} (E1) & x-y+z &=& 5 \\ (E2) & y-\frac{1}{2}z &=& -3 \\ (E3) & -z &=& -6 \end{cases}$$

Finally, we apply the second move from Theorem 8.1 one last time and multiply $E3$ by $-1$ to satisfy the conditions of Definition 8.3 for the variable $z$.

$$\begin{cases} (E1) & x-y+z &=& 5 \\ (E2) & y-\frac{1}{2}z &=& -3 \\ (E3) & -z &=& -6 \end{cases} \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \begin{cases} (E1) & x-y+z &=& 5 \\ (E2) & y-\frac{1}{2}z &=& -3 \\ (E3) & z &=& 6 \end{cases}$$

Now we proceed to substitute. Plugging in $z = 6$ into $E2$ gives $y - 3 = -3$ so that $y = 0$. With $y = 0$ and $z = 6$, $E1$ becomes $x - 0 + 6 = 5$, or $x = -1$. Our solution is $(-1, 0, 6)$. We leave it to the reader to check that substituting the respective values for $x$, $y$, and $z$ into the original system results in three identities. Since we have found a solution, the system is consistent; since there are no free variables, it is independent.

2. Proceeding as we did in 1, our first step is to get an equation with $x$ in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\begin{cases} (E1) & 2x+3y-z &=& 1 \\ (E2) & 10x-z &=& 2 \\ (E3) & 4x-9y+2z &=& 5 \end{cases} \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & 10x-z &=& 2 \\ (E3) & 4x-9y+2z &=& 5 \end{cases}$$

Now it's time to take care of the $x$'s in $E2$ and $E3$.

$$\begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & 10x-z &=& 2 \\ (E3) & 4x-9y+2z &=& 5 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -4E1+E3]{\text{Replace } E2 \text{ with } -10E1+E2} \begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & -15y+4z &=& -3 \\ (E3) & -15y+4z &=& 3 \end{cases}$$

Our next step is to get the coefficient of $y$ in $E2$ equal to 1. To that end, we have

$$\begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & -15y+4z &=& -3 \\ (E3) & -15y+4z &=& 3 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & y-\frac{4}{15}z &=& \frac{1}{5} \\ (E3) & -15y+4z &=& 3 \end{cases}$$

Finally, we rid $E3$ of $y$.

$$\begin{cases} (E1) & x+\frac{3}{2}y-\frac{1}{2}z &=& \frac{1}{2} \\ (E2) & y-\frac{4}{15}z &=& \frac{1}{5} \\ (E3) & -15y+4z &=& 3 \end{cases} \xrightarrow{\text{Replace } E3 \text{ with } 15E2+E3} \begin{cases} (E1) & x-y+z &=& 5 \\ (E2) & y-\frac{1}{2}z &=& -3 \\ (E3) & 0 &=& 6 \end{cases}$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 8.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on $x_1$.

$$\begin{cases} (E1) & 3x_1+x_2+x_4 &=& 6 \\ (E2) & 2x_1+x_2-x_3 &=& 4 \\ (E3) & x_2-3x_3-2x_4 &=& 0 \end{cases} \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \begin{cases} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 &=& 2 \\ (E2) & 2x_1+x_2-x_3 &=& 4 \\ (E3) & x_2-3x_3-2x_4 &=& 0 \end{cases}$$

Next we eliminate $x_1$ from $E2$

$$\begin{cases} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 &=& 2 \\ (E2) & 2x_1+x_2-x_3 &=& 4 \\ (E3) & x_2-3x_3-2x_4 &=& 0 \end{cases} \xrightarrow[\text{with } -2E1+E2]{\text{Replace } E2} \begin{cases} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 &=& 2 \\ (E2) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 &=& 0 \\ (E3) & x_2-3x_3-2x_4 &=& 0 \end{cases}$$

We switch $E2$ and $E3$ to get a coefficient of 1 for $x_2$.

$$\begin{cases} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 &=& 2 \\ (E2) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 &=& 0 \\ (E3) & x_2-3x_3-2x_4 &=& 0 \end{cases} \xrightarrow{\text{Switch } E2 \text{ and } E3} \begin{cases} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 &=& 2 \\ (E2) & x_2-3x_3-2x_4 &=& 0 \\ (E3) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 &=& 0 \end{cases}$$

Finally, we eliminate $x_2$ in $E3$.

$$\left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & x_2-3x_3-2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -\frac{1}{3}E2+E3]{\text{Replace } E3} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & x_2-3x_3-2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array}\right.$$

Equation $E3$ reduces to $0 = 0$,which is always true. Since we have no equations with $x_3$ or $x_4$ as leading variables, they are both free, which means we have a consistent dependent system. We parametrize the solution set by letting $x_3 = s$ and $x_4 = t$ and obtain from $E2$ that $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2-s-t$. Our solution is the set $\{(2-s-t, 2s+3t, s, t) : -\infty < s, t < \infty\}$.[13] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system. □

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving 2 above, it is clear after we eliminated the $x$'s in the second step to get the system

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

that equations $E2$ and $E3$ when taken together form a contradiction since we have identical left hand sides and different right hand sides. The algorithm takes two more steps to reach this contradiction. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, thus it gets called **back-substitution**. This may also be inefficient in many cases. Rest assured, the technique of substitution as you may have learned it in Intermediate Algebra will once again take center stage in Section 8.4. Lastly, we note that the system in 3 above is underdetermined, and as it is consistent, we have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations.

Example 8.1.3. Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

Solution. We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call $x$), the amount (in mL) of the 90% stock solution (which we'll call $y$) and the amount

[13] Here, any choice of $s$ and $t$ will determine a solution which is a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

(in mL) of water (which we'll call $w$). We now need to determine some relationships between these variables. Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions and the water. That is

$$\text{amount of 30\% stock solution} + \text{amount of 90\% stock solution} + \text{amount of water} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x + y + w = 500$. Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 90\% stock solution} = 200\,\text{mL}$$

The amount of acid in $x$ mL of 30% stock is $0.30x$ and the amount of acid in $y$ mL of 90% solution is $0.90y$. We have $0.30x + 0.90y = 200$. Converting to fractions,[14] our system of equations becomes

$$\begin{cases} x + y + w &= 500 \\ \frac{3}{10}x + \frac{9}{10}y &= 200 \end{cases}$$

We first eliminate the $x$ from the second equation

$$\begin{cases} (E1) \quad x + y + w &= 500 \\ (E2) \quad \frac{3}{10}x + \frac{9}{10}y &= 200 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1 + E2} \begin{cases} (E1) \quad x + y + w &= 500 \\ (E2) \quad \frac{3}{5}y - \frac{3}{10}w &= 50 \end{cases}$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\begin{cases} (E1) \quad x + y + w &= 500 \\ (E2) \quad \frac{3}{5}y - \frac{3}{10}w &= 50 \end{cases} \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \begin{cases} (E1) \quad x + y + w &= 500 \\ (E2) \quad y - \frac{1}{2}w &= \frac{250}{3} \end{cases}$$

Notice that we have no equation to determine $w$, and as such, $w$ is free. We set $w = t$ and from $E2$ get $y = \frac{1}{2}t + \frac{250}{3}$. Substituting into $E1$ gives $x + \left(\frac{1}{2}t + \frac{250}{3}\right) + t = 500$ so that $x = -\frac{3}{2}t + \frac{1250}{3}$. This system is consistent, dependent and its solution set is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) : -\infty < t < \infty\right\}$. While this answer checks algebraically, we have neglected to take into account that $x$, $y$ and $w$, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$. The constraint $x \geq 0$ gives us $-\frac{3}{2}t + \frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t + \frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. Our final answer is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) : 0 \leq t \leq \frac{2500}{9}\right\}$. Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y = 100$

[14] We do this only because we believe students can use all of the practice with fractions they can get!

so that $\frac{1}{2}t + \frac{250}{3} = 100$, and we get $t = \frac{100}{3}$. This means the amount of 30% solution required is $x = -\frac{3}{2}t + \frac{1250}{3} = -\frac{3}{2}\left(\frac{100}{3}\right) + \frac{1250}{3} = \frac{1100}{3}$ mL, and for the water, $w = t = \frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. □

### 8.1.1 EXERCISES

1. Put the following systems of linear equations into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

(a) $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

(b) $\begin{cases} x + y + z = 3 \\ 2x - y + z = 0 \\ -3x + 5y + 7z = 7 \end{cases}$

(c) $\begin{cases} 4x - y + z = 5 \\ 2y + 6z = 30 \\ x + z = 5 \end{cases}$

(d) $\begin{cases} 4x - y + z = 5 \\ 2y + 6z = 30 \\ x + z = 6 \end{cases}$

(e) $\begin{cases} x + y + z = -17 \\ y - 3z = 0 \end{cases}$

(f) $\begin{cases} x - 2y + 3z = 7 \\ -3x + y + 2z = -5 \\ 2x + 2y + z = 3 \end{cases}$

(g) $\begin{cases} 3x - 2y + z = -5 \\ x + 3y - z = 12 \\ x + y + 2z = 0 \end{cases}$

(h) $\begin{cases} 2x - y + z = -1 \\ 4x + 3y + 5z = 1 \\ 5y + 3z = 4 \end{cases}$

(i) $\begin{cases} x - y + z = -4 \\ -3x + 2y + 4z = -5 \\ x - 5y + 2z = -18 \end{cases}$

(j) $\begin{cases} 2x - 4y + z = -7 \\ x - 2y + 2z = -2 \\ -x + 4y - 2z = 3 \end{cases}$

(k) $\begin{cases} 2x - y + z = 1 \\ 2x + 2y - z = 1 \\ 3x + 6y + 4z = 9 \end{cases}$

(l) $\begin{cases} x - 3y - 4z = 3 \\ 3x + 4y - z = 13 \\ 2x - 19y - 19z = 2 \end{cases}$

(m) $\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y = 2 \end{cases}$

(n) $\begin{cases} x - y + z = 8 \\ 3x + 3y - 9z = -6 \\ 7x - 2y + 5z = 39 \end{cases}$

(o) $\begin{cases} 2x - 3y + z = -1 \\ 4x - 4y + 4z = -13 \\ 6x - 5y + 7z = -25 \end{cases}$

(p)
$$\begin{cases} 2x_1 + x_2 - 12x_3 - x_4 = 16 \\ -x_1 + x_2 + 12x_3 - 4x_4 = -5 \\ 3x_1 + 2x_2 - 16x_3 - 3x_4 = 25 \\ x_1 + 2x_2 - 5x_4 = 11 \end{cases}$$

(r)
$$\begin{cases} x_1 - x_2 - 5x_3 + 3x_4 = -1 \\ x_1 + x_2 + 5x_3 - 3x_4 = 0 \\ x_2 + 5x_3 - 3x_4 = 1 \\ x_1 - 2x_2 - 10x_3 + 6x_4 = -1 \end{cases}$$

(q)
$$\begin{cases} x_1 - x_3 = -2 \\ 2x_2 - x_4 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ -x_3 + x_4 = 1 \end{cases}$$

2. Find two other forms of the parametric solution to Exercise 1c above by reorganizing the equations so that $x$ or $y$ can be the free variable.

3. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

4. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

5. Discuss with your classmates how you would approach Exercise 4 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

6. Discuss with your classmates why it is impossible to mix a 20% acid solution with a 40% acid solution to produce a 60% acid solution. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like?

### 8.1.2 ANSWERS

1. Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

(a) $\begin{cases} x+y &= 5 \\ y &= 7 \end{cases}$ Consistent independent
Solution $(-2, 7)$

(b) $\begin{cases} x-\frac{5}{3}y-\frac{7}{3}z &= -\frac{7}{3} \\ y+\frac{5}{4}z &= 2 \\ z &= 0 \end{cases}$ Consistent independent
Solution $(1, 2, 0)$

(c) $\begin{cases} x-\frac{1}{4}y+\frac{1}{4}z &= \frac{5}{4} \\ y+3z &= 15 \\ 0 &= 0 \end{cases}$ Consistent dependent
Solution $(-t+5, -3t+15, t)$
for all real numbers $t$

(d) $\begin{cases} x-\frac{1}{4}y+\frac{1}{4}z &= \frac{5}{4} \\ y+3z &= 15 \\ 0 &= 1 \end{cases}$ Inconsistent
No solution

(e) $\begin{cases} x+y+z &= -17 \\ y-3z &= 0 \end{cases}$ Consistent dependent
Solution $(-4t-17, 3t, t)$
for all real numbers $t$

(f) $\begin{cases} x-2y+3z &= 7 \\ y-\frac{11}{5}z &= -\frac{16}{5} \\ z &= 1 \end{cases}$ Consistent independent
Solution $(2, -1, 1)$

(g) $\begin{cases} x+y+2z &= 0 \\ y-\frac{3}{2}z &= 6 \\ z &= -2 \end{cases}$ Consistent independent
Solution $(1, 3, -2)$

(h) $\begin{cases} x-\frac{1}{2}y+\frac{1}{2}z &= -\frac{1}{2} \\ y+\frac{3}{5}z &= \frac{3}{5} \\ 0 &= 1 \end{cases}$ Inconsistent
no solution

(i) $\begin{cases} x - y + z = -4 \\ y - 7z = 17 \\ z = -2 \end{cases}$ Consistent independent
Solution $(1, 3, -2)$

(j) $\begin{cases} x - 2y + 2z = -2 \\ y = \frac{1}{2} \\ z = 1 \end{cases}$ Consistent independent
Solution $\left(-3, \frac{1}{2}, 1\right)$

(k) $\begin{cases} x - \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2} \\ y - \frac{2}{3}z = 0 \\ z = 1 \end{cases}$ Consistent independent
Solution $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

(l) $\begin{cases} x - 3y - 4z = 3 \\ y + \frac{11}{13}z = \frac{4}{13} \\ 0 = 0 \end{cases}$ Consistent dependent
Solution $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
for all real numbers $t$

(m) $\begin{cases} x + y + z = 4 \\ y + \frac{1}{2}z = \frac{3}{2} \\ 0 = 1 \end{cases}$ Inconsistent
no solution

(n) $\begin{cases} x - y + z = 8 \\ y - 2z = -5 \\ z = 1 \end{cases}$ Consistent independent
Solution $(4, -3, 1)$

(o) $\begin{cases} x - \frac{3}{2}y + \frac{1}{2}z = -\frac{1}{2} \\ y + z = -\frac{11}{2} \\ 0 = 0 \end{cases}$ Consistent dependent
Solution $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
for all real numbers $t$

(p) $\begin{cases} x_1 + \frac{2}{3}x_2 - \frac{16}{3}x_3 - x_4 = \frac{25}{3} \\ x_2 + 4x_3 - 3x_4 = 2 \\ 0 = 0 \\ 0 = 0 \end{cases}$ Consistent dependent
Solution $(8s - t + 7, -4s + 3t + 2, s, t)$
for all real numbers $s$ and $t$

(q)
$$\begin{cases} x_1 - x_3 = -2 \\ x_2 - \frac{1}{2}x_4 = 0 \\ x_3 - \frac{1}{2}x_4 = 1 \\ x_4 = 4 \end{cases}$$
Consistent independent
Solution $(1, 2, 3, 4)$

(r)
$$\begin{cases} x_1 - x_2 - 5x_3 + 3x_4 = -1 \\ x_2 + 5x_3 - 3x_4 = \frac{1}{2} \\ 0 = 1 \\ 0 = 0 \end{cases}$$
Inconsistent
No solution

2. If $x$ is the free variable then the solution is $(t, 3t, -t+5)$ and if $y$ is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t + 5\right)$.

3. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.

4. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and $t$ pounds of Type III where $0 \leq t \leq \frac{4}{3}$.

## 8.2 Systems of Linear Equations: Augmented Matrices*

In Section 8.1 we introduced Gaussian Elimination as a means of transforming a system of linear equations into triangular form with the ultimate goal of producing an equivalent system of linear equations which is easier to solve. If take a step back and study the process, we see that all of our moves are determined entirely by the *coefficients* of the variables involved, and not the variables themselves. Much the same thing happened when we studied long division in Section 4.2. Just as we developed synthetic division to streamline that process, in this section, we introduce a similar bookkeeping device to help us solve systems of linear equations. To that end, we define a **matrix** as a rectangular array of real numbers. We typically enclose matrices with square brackets, '[' and ']', and we size matrices by the number of rows and columns they have. For example, the **size** (sometimes called the **dimension**) of

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

is $2 \times 3$ because it has 2 rows and 3 columns. The individual numbers in a matrix are called its **entries** and are usually labeled with double subscripts: the first tells which row the element is in and the second tells which column it is in. The rows are numbered from top to bottom and the columns are numbered from left to right. Matrices themselves are usually denoted by uppercase letters ($A$, $B$, $C$, etc.) while their entries are usually denoted by the corresponding letter. So, for instance, if we have

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -5 & 10 \end{bmatrix}$$

then $a_{11} = 3$, $a_{12} = 0$, $a_{13} = -1$, $a_{21} = 2$, $a_{22} = -5$, and $a_{23} = 10$. We shall explore matrices as mathematical objects with their own algebra in Section **??** and introduce them here solely as a bookkeeping device. Consider the system of linear equations from number 2 in Example 8.1.2

$$\begin{cases} (E1) & 2x + 3y - z &= 1 \\ (E2) & 10x - z &= 2 \\ (E3) & 4x - 9y + 2z &= 5 \end{cases}$$

We encode this system into a matrix by assigning each equation to a corresponding row. Within that row, each variable and the constant gets its own column, and to separate the variables on the left hand side of the equation from the constants on the right hand side, we use a vertical bar, |. Note that in $E2$, since $y$ is not present, we record its coefficient as 0. The matrix associated with

this system is

$$\begin{array}{r} \\ (E1) \to \\ (E2) \to \\ (E3) \to \end{array} \left[\begin{array}{rrr|r} x & y & z & c \\ \hline 2 & 3 & -1 & 1 \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right]$$

This matrix is called an **augmented matrix** because the column containing the constants is appended to the matrix containing the coefficients.[1] To solve this system, we can use the same kind operations on the *rows* of the matrix that we performed on the *equations* of the system. More specifically, we have the following analog of Theorem 8.1 below.

THEOREM 8.2. **Row Operations:** Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b] Where we add entries in corresponding columns.

As a demonstration of the moves in Theorem 8.2, we revisit some of the steps that were used in solving the systems of linear equations in Example 8.1.2 of Section 8.1. The reader is encouraged to perform the indicated operations on the rows of the augmented matrix to see that the machinations are identical to what is done to the coefficients of the variables in the equations. We first see a demonstration of switching two rows using the first step of part 1 in Example 8.1.2.

$$\left\{\begin{array}{lrcr} (E1) & 3x - y + z & = & 3 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & x - y + z & = & 5 \end{array}\right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array}\right.$$

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R3} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

[1] We shall study the coefficient and constant matrices separately in Section **??**.

Next, we have a demonstration of replacing a row with a nonzero multiple of itself using the first step of part 3 in Example 8.1.2.

$$\begin{cases} (E1) & 3x_1 + x_2 + x_4 &= 6 \\ (E2) & 2x_1 + x_2 - x_3 &= 4 \\ (E3) & x_2 - 3x_3 - 2x_4 &= 0 \end{cases} \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \begin{cases} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 &= 2 \\ (E2) & 2x_1 + x_2 - x_3 &= 4 \\ (E3) & x_2 - 3x_3 - 2x_4 &= 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 1 & 0 & 1 & 6 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R1} \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 2 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right]$$

Finally, we have an example of replacing a row with itself plus a multiple of another row using the second step from part 2 in Example 8.1.2.

$$\begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z &= \frac{1}{2} \\ (E2) & 10x - z &= 2 \\ (E3) & 4x - 9y + 2z &= 5 \end{cases} \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \begin{cases} (E1) & x + \frac{3}{2}y - \frac{1}{2}z &= \frac{1}{2} \\ (E2) & -15y + 4z &= -3 \\ (E3) & -15y + 4z &= 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -4R1 + R3]{\text{Replace } R2 \text{ with } -10R1 + R2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -15 & 4 & -3 \\ 0 & -15 & 4 & 3 \end{array}\right]$$

The matrix equivalent of 'triangular form' is **row echelon form**. The reader is encouraged to refer to Definition 8.3 for comparison. Note that the analog of 'leading variable' of an equation is 'leading entry' of a row. Specifically, the first nonzero entry (if it exists) in a row is called the **leading entry** of that row.

Definition 8.4. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

To solve a system of a linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to the rows to get the matrix into row-echelon form. We then decode the matrix and back substitute. The next example illustrates this nicely.

EXAMPLE 8.2.1. Use an augmented matrix to transform the following system of linear equations into triangular form. Solve the system.

$$\begin{cases} 3x - y + z &= 8 \\ x + 2y - z &= 4 \\ 2x + 3y - 4z &= 10 \end{cases}$$

SOLUTION. We first encode the system into an augmented matrix.

$$\begin{cases} 3x - y + z &= 8 \\ x + 2y - z &= 4 \\ 2x + 3y - 4z &= 10 \end{cases} \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Thinking back to Gaussian Elimination at an equations level, our first order of business is to get $x$ in $E1$ with a coefficient of 1. At the matrix level, this means getting a leading 1 in $R1$. This is in accordance with the first criteria in Definition 8.4. To that end, we interchange $R1$ and $R2$.

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right]$$

Our next step is to eliminate the $x$'s from $E2$ and $E3$. From a matrix standpoint, this means we need 0's below the leading 1 in $R1$. This guarantees the leading 1 in $R2$ will be to the right of the leading 1 in $R1$ in accordance with the second requirement of Definition 8.4.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -2R1 + R3]{\text{Replace } R2 \text{ with } -3R1 + R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right]$$

Now we repeat the above process for the variable $y$ which means we need to get the leading entry in $R2$ to be 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -\frac{1}{7}R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right]$$

To guarantee the leading 1 in $R3$ is to the right of the leading 1 in $R2$, we get a 0 in the second column of $R3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } R2+R3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right]$$

Finally, we get the leading entry in $R3$ to be 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{7}{18}R3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Decoding from the matrix gives a system in triangular form

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcl} x+2y-z & = & 4 \\ y-\frac{4}{7}z & = & \frac{4}{7} \\ z & = & -1 \end{array}\right.$$

We get $z=-1$, $y=\frac{4}{7}z+\frac{4}{7}=\frac{4}{7}(-1)+\frac{4}{7}=0$ and $x=-2y+z+4=-2(0)+(-1)+4=3$ for a final answer of $(3,0,-1)$. We leave it to the reader to check. □

As part of Gaussian Elimination, we used row operations to obtain 0's beneath each leading 1 to put the matrix into row echelon form. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

Definition 8.5. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1s are the only nonzero entry in their respective columns.

Of what significance is the reduced row echelon form of a matrix? To illustrate, let's take the row echelon form from Example 8.2.1 and perform the necessary steps to put into reduced row echelon form. We start by using the leading 1 in $R3$ to zero out the numbers in the rows above it.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\text{Replace } R2 \text{ with } \frac{4}{7}R3+R2]{\text{Replace } R1 \text{ with } R3+R1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Finally, we take care of the 2 in $R1$ above the leading 1 in $R2$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -2R2+R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To our surprise and delight, when we decode this matrix, we obtain the solution instantly without having to deal with any back-substitution at all.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \begin{cases} x &= 3 \\ y &= 0 \\ z &= -1 \end{cases}$$

Note that in the previous discussion, we could have started with $R2$ and used it to get a zero above its leading 1 and then done the same for the leading 1 in $R3$. By starting with $R3$, however, we get more zeros first, and the more zeros there are, the faster the remaining calculations will be.[2] It is also worth noting that while a matrix has several[3] row echelon forms, it has only one reduced row echelon form. The process by which we have put a matrix into reduced row echelon form is called **Gauss-Jordan Elimination**.

EXAMPLE 8.2.2. Solve the following system using an augmented matrix. Use Gauss-Jordan Elimination to put the augmented matrix into reduced row echelon form.

$$\begin{cases} x_2 - 3x_1 + x_4 &= 2 \\ 2x_1 + 4x_3 &= 5 \\ 4x_2 - x_4 &= 3 \end{cases}$$

SOLUTION. We first encode the system into a matrix. (Pay attention to the subscripts!)

$$\begin{cases} x_2 - 3x_1 + x_4 &= 2 \\ 2x_1 + 4x_3 &= 5 \\ 4x_2 - x_4 &= 3 \end{cases} \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{cccc|c} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

[2] Carl also finds starting with $R3$ to be more symmetric, in a purely poetic way.
[3] infinite, in fact

Next, we get a leading 1 in the first column of $R1$.

$$\left[\begin{array}{cccc|c} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -\frac{1}{3}R1} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

Now we eliminate the nonzero entry below our leading 1.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -2R1+R2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We proceed to get a leading 1 in $R2$.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } \frac{3}{2}R2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We now zero out the entry below the leading 1 in $R2$.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -4R2+R3} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right]$$

Next, it's time for a leading 1 in $R3$.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{1}{24}R3} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

The matrix is now in row echelon form. To get the reduced row echelon form, we start with the last leading 1 we produced and work to get 0's above it.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -6R3+R2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

Lastly, we get a 0 above the leading 1 of $R2$.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R2 + R1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

At last, we decode to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x_1 - \frac{5}{12}x_4 & = & -\frac{5}{12} \\ x_2 - \frac{1}{4}x_4 & = & \frac{3}{4} \\ x_3 + \frac{5}{24}x_4 & = & \frac{35}{24} \end{array}\right.$$

We have that $x_4$ is free and we assign it the parameter $t$. We obtain $x_3 = -\frac{5}{24}t + \frac{35}{24}$, $x_2 = \frac{1}{4}t + \frac{3}{4}$, and $x_1 = \frac{5}{12}t - \frac{5}{12}$. Our solution is $\left\{\left(\frac{5}{12}t - \frac{5}{12}, \frac{1}{4}t + \frac{3}{4}, -\frac{5}{24}t + \frac{35}{24}, t\right) : -\infty < t < \infty\right\}$ and leave it to the reader to check. □

Like all good algorithms, putting a matrix in row echelon or reduced row echelon form can easily be programmed into a calculator, and, doubtless, your graphing calculator has such a feature. We use this in our next example.

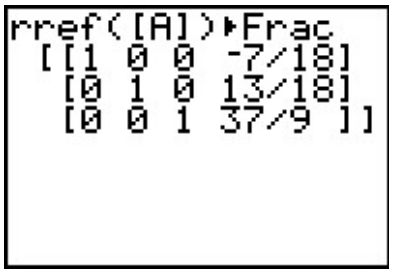
EXAMPLE 8.2.3. Find the quadratic function which passes through the points $(-1, 3)$, $(2, 4)$, $(5, -2)$.

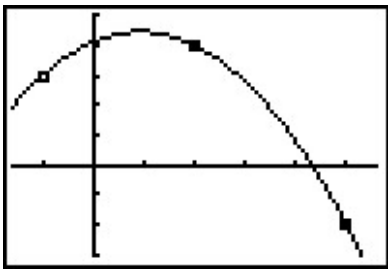
SOLUTION. According to Definition 3.4, a quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. Our goal is to find $a$, $b$ and $c$ so that the three given points are on the graph of $f$. If $(-1, 3)$ is on the graph of $f$, then $f(-1) = 3$, or $a(-1)^2 + b(-1) + c = 3$ which reduces to $a - b + c = 3$, an honest-to-goodness linear equation with the variables $a$, $b$ and $c$. Since the point $(2, 4)$ is also on the graph of $f$, then $f(2) = 4$ which gives us the equation $4a + 2b + c = 4$. Lastly, the point $(5, -2)$ is on the graph of $f$ gives us $25a + 5b + c = -2$. Putting these together, we obtain a system of three linear equations. Encoding this into an augmented matrix produces

$$\left\{\begin{array}{rcr} a - b + c & = & 3 \\ 4a + 2b + c & = & 4 \\ 25a + 5b + c & = & -2 \end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 4 \\ 25 & 5 & 1 & -2 \end{array}\right]$$

Using a calculator,[4] we find $a = -\frac{7}{18}$, $b = \frac{13}{18}$ and $c = \frac{37}{9}$. Hence, the one and only quadratic which fits the bill is $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$. To verify this analytically, we see that $f(-1) = 3$, $f(2) = 4$, and $f(5) = -2$. We can use the calculator to check our solution as well by plotting the three data points and the function $f$.

[4] We've tortured you enough already with fractions in this exposition!



The graph of $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$
with the points $(-1, 3)$, $(2, 4)$ and $(5, -2)$

□

### 8.2.1 EXERCISES

1. State whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

(a) $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

2. The following matrices are in reduced row echelon form. Decode from each matrix the solution of the corresponding system of linear equations or state that the system is inconsistent.

(a) $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

(e) $\left[\begin{array}{cccc|c} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(f) $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(g) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

3. Solve the following systems of linear equations using the techniques discussed in this section. Compare and contrast these techniques with those you used to solve the systems in the Exercises in Section 8.1.

(a) $\begin{cases} -5x + y = 17 \\ x + y = 5 \end{cases}$

(b) $\begin{cases} x + y + z = 3 \\ 2x - y + z = 0 \\ -3x + 5y + 7z = 7 \end{cases}$

(c) $\begin{cases} 4x - y + z = 5 \\ 2y + 6z = 30 \\ x + z = 5 \end{cases}$

(d)
$$\begin{cases} x - 2y + 3z = 7 \\ -3x + y + 2z = -5 \\ 2x + 2y + z = 3 \end{cases}$$

(e)
$$\begin{cases} 3x - 2y + z = -5 \\ x + 3y - z = 12 \\ x + y + 2z = 0 \end{cases}$$

(f)
$$\begin{cases} 2x - y + z = -1 \\ 4x + 3y + 5z = 1 \\ 5y + 3z = 4 \end{cases}$$

(g)
$$\begin{cases} x - y + z = -4 \\ -3x + 2y + 4z = -5 \\ x - 5y + 2z = -18 \end{cases}$$

(h)
$$\begin{cases} 2x - 4y + z = -7 \\ x - 2y + 2z = -2 \\ -x + 4y - 2z = 3 \end{cases}$$

(i)
$$\begin{cases} 2x - y + z = 1 \\ 2x + 2y - z = 1 \\ 3x + 6y + 4z = 9 \end{cases}$$

(j)
$$\begin{cases} x - 3y - 4z = 3 \\ 3x + 4y - z = 13 \\ 2x - 19y - 19z = 2 \end{cases}$$

(k)
$$\begin{cases} x + y + z = 4 \\ 2x - 4y - z = -1 \\ x - y = 2 \end{cases}$$

(l)
$$\begin{cases} x - y + z = 8 \\ 3x + 3y - 9z = -6 \\ 7x - 2y + 5z = 39 \end{cases}$$

(m)
$$\begin{cases} 2x - 3y + z = -1 \\ 4x - 4y + 4z = -13 \\ 6x - 5y + 7z = -25 \end{cases}$$

(n)
$$\begin{cases} x_1 - x_3 = -2 \\ 2x_2 - x_4 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ -x_3 + x_4 = 1 \end{cases}$$

4. The price for admission into the Stitz-Zeager Sasquatch Museum and Research Station is \$15 for adults and \$8 for kids 13 years old and younger. When the Zahlenreich family visits the museum their bill is \$38 and when the Nullsatz family visits their bill is \$39. One day both families went together and took an adult babysitter along to watch the kids and the total admission charge was \$92. Later that summer, the adults from both families went without the kids and the bill was \$45. Is that enough information to determine how many adults and children are in each family? If not, state whether the resulting system is inconsistent or consistent dependent. In the latter case, give at least two plausible solutions.

5. Use the technique in Example 8.2.3 to find the line between the points $(-3, 4)$ and $(6, 1)$. How does your answer compare to the slope-intercept form of the line in Equation 3.3?

6. With the help of your classmates, find at least two different row echelon forms for the matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$$

### 8.2.2 Answers

1. (a) Reduced row echelon form
   (b) Neither
   (c) Row echelon form only
   (d) Reduced row echelon form
   (e) Reduced row echelon form
   (f) Row echelon form only

2. (a) $(-2, 7)$
   (b) $(-3, 20, 19)$
   (c) $(-3t + 4, -6t - 6, 2, t)$ for all real numbers $t$
   (d) $(2, -1, 1)$
   (e) $(1, 3, -2)$
   (f) Inconsistent
   (g) $(1, 3, -2)$
   (h) $\left(-3, \frac{1}{2}, 1\right)$
   (i) $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
   (j) $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$ for all real numbers $t$
   (k) Inconsistent
   (l) $(4, -3, 1)$
   (m) $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
   (n) Inconsistent
   (o) $(8s - t + 7, -4s + 3t + 2, s, t)$ for all real numbers $s$ and $t$
   (p) $(-9t - 3, 4t + 20, t)$ for all real numbers $t$
   (q) Inconsistent

3. (a) $(-2, 7)$
   (b) $(1, 2, 0)$
   (c) $(-t + 5, -3t + 15, t)$ for all real numbers $t$
   (d) $(1, 2, 3, 4)$

4. Let $x_1$ and $x_2$ be the numbers of adults and children, respectively, in the Zahlenreich family and let $x_3$ and $x_4$ be the numbers of adults and children, respectively, in the Nullsatz family. The system of equations determined by the given information is

$$\begin{cases} 15x_1 + 8x_2 = 38 \\ 15x_3 + 8x_4 = 39 \\ 15x_1 + 8x_2 + 15x_3 + 8x_4 = 77 \\ 15x_1 + 15x_3 = 45 \end{cases}$$

   We subtracted the cost of the babysitter in E3 so the constant is 77, not 92. This system is consistent dependent and its solution is $\left(\frac{8}{15}t + \frac{2}{5}, -t + 4, -\frac{8}{15}t + \frac{13}{5}, t\right)$. Our variables represent numbers of adults and children so they must be whole numbers. Running through the values $t = 0, 1, 2, 3, 4$ yields only one solution where all four variables are whole numbers; $t = 3$ gives us $(2, 1, 1, 3)$. Thus there are 2 adults and 1 child in the Zahlenreichs and 1 adult and 3 kids in the Nullsatzs.

## 8.3 DETERMINANTS AND CRAMER'S RULE*

### 8.3.1 DEFINITION AND PROPERTIES OF THE DETERMINANT

In this section we assign to each square matrix $A$ a real number, called the **determinant** of $A$, which will eventually lead us to yet another technique for solving consistent independent systems of linear equations. The determinant is defined recursively, that is, we define it for $1 \times 1$ matrices and give a rule by which we can reduce determinants of $n \times n$ matrices to a sum of determinants of $(n-1) \times (n-1)$ matrices.[1] This means we will be able to evaluate the determinant of a $2 \times 2$ matrix as a sum of the determinants of $1 \times 1$ matrices; the determinant of a $3 \times 3$ matrix as a sum of the determinants of $2 \times 2$ matrices, and so forth. To explain how we will take an $n \times n$ matrix and distill from it an $(n-1) \times (n-1)$, we use the following notation.

DEFINITION 8.6. Given an $n \times n$ matrix $A$ where $n > 1$, the matrix $A_{ij}$ is the $(n-1) \times (n-1)$ matrix formed by deleting the $i$th row of $A$ and the $j$th column of $A$.

For example, using the matrix $A$ below, we find the matrix $A_{23}$ by deleting the second row and third column of $A$.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Delete } R2 \text{ and } C3} A_{23} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

We are now in the position to define the determinant of a matrix.

DEFINITION 8.7. Given an $n \times n$ matrix $A$ the **determinant of $A$**, denoted $\det(A)$, is defined as follows

- If $n = 1$, then $A = [a_{11}]$ and $\det(A) = \det([a_{11}]) = a_{11}$.
- If $n > 1$, then $A = [a_{ij}]_{n \times n}$ and

$$\det(A) = \det\left([a_{ij}]_{n \times n}\right) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

There are two commonly used notations for the determinant of a matrix $A$: '$\det(A)$' and '$|A|$' We have chosen to use the notation $\det(A)$ as opposed to $|A|$ because we find that the latter is often confused with absolute value, especially in the context of a $1 \times 1$ matrix. In the expansion

[1] We will talk more about the term 'recursively' in Section 9.1.

$a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n}a_{1n}\det(A_{1n})$, the notation '$+ - \ldots + (-1)^{1+n}a_{1n}$' means that the signs alternate and the final sign is dictated by the sign of the quantity $(-1)^{1+n}$. Since the entries $a_{11}$, $a_{12}$ and so forth up through $a_{1n}$ comprise the first row of $A$, we say we are finding the determinant of $A$ by 'expanding along the first row'. Later in the section, we will develop a formula for $\det(A)$ which allows us to find it by expanding along any row.

Applying Definition 8.7 to the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ we get

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\right) \\ &= 4\det(A_{11}) - (-3)\det(A_{12}) \\ &= 4\det([1]) + 3\det([2]) \\ &= 4(1) + 3(2) \\ &= 10 \end{aligned}$$

For a generic $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= a\det(A_{11}) - b\det(A_{12}) \\ &= a\det([d]) - b\det([c]) \\ &= ad - bc \end{aligned}$$

This formula is worth remembering

Equation 8.1. For a $2 \times 2$ matrix,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Applying Definition 8.7 to the $3 \times 3$ matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ we obtain

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) \\
&= 3\det(A_{11}) - 1\det(A_{12}) + 2\det(A_{13}) \\
&= 3\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\right) \\
&= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) \\
&= 3(-9) - (-10) + 2(2) \\
&= -13
\end{aligned}$$

To evaluate the determinant of a $4 \times 4$ matrix, we would have to evaluate the determinants of *four* $3 \times 3$ matrices, each of which involves the finding the determinants of *three* $2 \times 2$ matrices.

### 8.3.2 CRAMER'S RULE

In this section, we introduce a theorem which enables us to solve a system of linear equations by means of determinants only. As usual, the theorem is stated in full generality, using numbered unknowns $x_1$, $x_2$, etc., instead of the more familiar letters $x$, $y$, $z$, etc. The proof of the general case is best left to a course in Linear Algebra.

THEOREM 8.3. **Cramer's Rule:** Suppose $AX = B$ is the matrix form of a system of $n$ linear equations in $n$ unknowns where $A$ is the coefficient matrix, $X$ is the unknowns matrix, and $B$ is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1$, $x_2$, $\ldots x_n$ is given by:

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where $A_j$ is the matrix $A$ whose $j$th column has been replaced by the constants in $B$.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of $A_j$ to that of the determinant of the coefficient matrix. The matrix $A_j$ is found by replacing the column in the coefficient matrix which holds the coefficients of $x_j$ with the constants of the system. The following example fleshes out this method.

EXAMPLE 8.3.1. Use Cramer's Rule to solve for the indicated unknowns.

1. Solve $\begin{cases} 2x_1 - 3x_2 &= 4 \\ 5x_1 + x_2 &= -2 \end{cases}$ for $x_1$ and $x_2$

2. Solve $\begin{cases} 2x - 3y + z &= -1 \\ x - y + z &= 1 \\ 3x - 4z &= 0 \end{cases}$ for $z$.

Solution.

1. Writing this system in matrix form, we find

$$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

   To find the matrix $A_1$, we remove the column of the coefficient matrix $A$ which holds the coefficients of $x_1$ and replace it with the corresponding entries in $B$. Likewise, we replace the column of $A$ which corresponds to the coefficients of $x_2$ with the constants to form the matrix $A_2$. This yields

$$A_1 = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 4 \\ 5 & -2 \end{bmatrix}$$

   Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$ and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

   The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

2. To use Cramer's Rule to find $z$, we identify $x_3$ as $z$. We have

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A_3 = A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

   Expanding both $\det(A)$ and $\det(A_z)$ along the third rows (to take advantage of the 0's) gives

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$$

The reader is encouraged to solve this system for $x$ and $y$ similarly and check the answer. $\square$

### 8.3.3 Exercises

1. Compute the determinant of the following matrices.

(a) $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

(b) $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

(c) $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

(d) $L = \begin{bmatrix} \frac{1}{x^3} & \frac{\ln(x)}{x^3} \\ -\frac{3}{x^4} & \frac{1-3\ln(x)}{x^4} \end{bmatrix}$

(e) $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

(f) $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

(g) $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

(h) $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

2. Use Cramer's Rule to solve the system of linear equations.

(a) $\begin{cases} 3x + 7y &= 26 \\ 5x + 12y &= 39 \end{cases}$

(b) $\begin{cases} x + y + z &= 3 \\ 2x - y + z &= 0 \\ -3x + 5y + 7z &= 7 \end{cases}$

3. Use Cramer's Rule to solve for $x_4$ in the following system of linear equations.

$$\begin{cases} x_1 - x_3 &= -2 \\ 2x_2 - x_4 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ -x_3 + x_4 &= 1 \end{cases}$$

### 8.3.4 ANSWERS

1. (a) $\det(B) = 1$
   (b) $\det(C) = 0$
   (c) $\det(Q) = x^2$
   (d) $\det(L) = \frac{1}{x^7}$
   (e) $\det(F) = -12$
   (f) $\det(G) = 0$
   (g) $\det(V) = 20i + 43j + 4k$
   (h) $\det(H) = -2$

2. (a) $x = 39,\ y = -13$
   (b) $x = 1,\ y = 2,\ z = 0$

## 8.4 SYSTEMS OF NON-LINEAR EQUATIONS AND INEQUALITIES

In this section, we study systems of non-linear equations and inequalities. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

EXAMPLE 8.4.1. Solve the following systems of equations. Verify your answers algebraically and graphically.

1. $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 + 9y^2 = 36 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 4 \\ 4x^2 - 9y^2 = 36 \end{cases}$
3. $\begin{cases} x^2 + y^2 = 4 \\ y - 2x = 0 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 4 \\ y - x^2 = 0 \end{cases}$

SOLUTION:

1. Since both equations contain $x^2$ and $y^2$ only, we can eliminate one of the variables as we did in Section 8.1.

$$\begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & 4x^2 + 9y^2 = 36 \end{cases} \xrightarrow[-4E1 + E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + y^2 = 4 \\ (E2) & 5y^2 = 20 \end{cases}$$

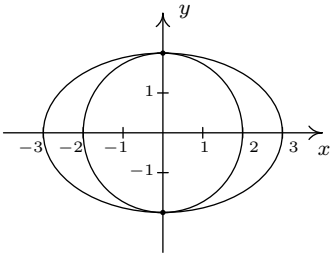
From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated $x$ values, we substitute each value of $y$ into one of the equations to find the resulting value of $x$. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0, 2), (0, -2)\}$.

To check this algebraically, we need to show that both points satisfy both of the original equations. We leave it to the reader to verify this. To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2 + 9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is easily recognized as an ellipse centered at $(0, 0)$ with a major axis along the $x$-axis of length 6 and a minor axis along the $y$-axis of length 4. We see from the graph that the two curves intersect at their $y$-intercepts only, $(0, \pm 2)$.

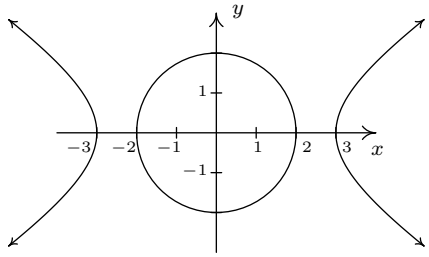
2. We proceed as before to eliminate one of the variables

$$\left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & 4x^2 - 9y^2 & = & 36 \end{array} \right. \xrightarrow[-4E1 + E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{lrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{array} \right.$$

Since the equation $-13y^2 = 20$ admits no real solution, the system is inconsistent. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. We see that the circle and the hyperbola have no points in common.




Graphs for $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 + 9y^2 & = & 36 \end{array} \right.$



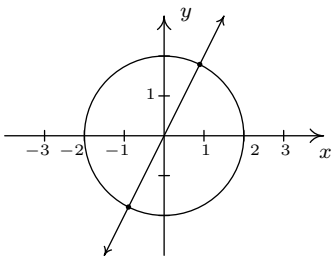

Graphs for $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 - 9y^2 & = & 36 \end{array} \right.$

3. Since there are no like terms among the two equations, elimination won't do us any good. We turn to substitution and from the equation $y - 2x = 0$, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Solving, we find $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$. Returning to the equation we used for the substitution, $y = 2x$, we find $y = \frac{4\sqrt{5}}{5}$ when $x = \frac{2\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$. Similarly, we find the other solution to be $\left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)$. We leave it to the reader that both points satisfy both equations, so that our final answer is $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)\right\}$. The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$ is a line through the origin with slope 2. Though we cannot verify the numerical values of the points of intersection from our sketch, we do see that we have two solutions: one in Quadrant I and one in Quadrant III as required.

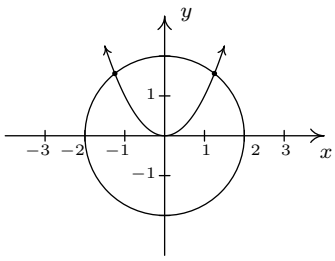
4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination.[1]

$$\begin{cases} (E1) & x^2 + y^2 &=& 4 \\ (E2) & y - x^2 &=& 0 \end{cases} \xrightarrow[E1 + E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + y^2 &=& 4 \\ (E2) & y^2 + y &=& 4 \end{cases}$$

From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$ which gives $y = \frac{-1 \pm \sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations to head off any extraneous solutions we may encounter. We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly twice, and both of these points have a positive $y$ value. Of the two solutions for $y$, only $y = \frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this into $y - x^2 = 0$ and solve for $x$. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$. Our solution is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$, which we leave to the reader to verify.




Graphs for $\begin{cases} x^2 + y^2 &=& 4 \\ y - 2x &=& 0 \end{cases}$




Graphs for $\begin{cases} x^2 + y^2 &=& 4 \\ y - x^2 &=& 36 \end{cases}$

□

A couple of remarks about Example 8.4.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'. Secondly, as we saw with number 4, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it sometimes pays to take advantage when they are. Our next example provides some considerable review of many of the topics introduced in this text.

EXAMPLE 8.4.2. Solve the following systems of equations. Verify your answers algebraically and graphically, as appropriate.

[1] We encourage the reader to solve the system using substitution to see that you get the same solution.

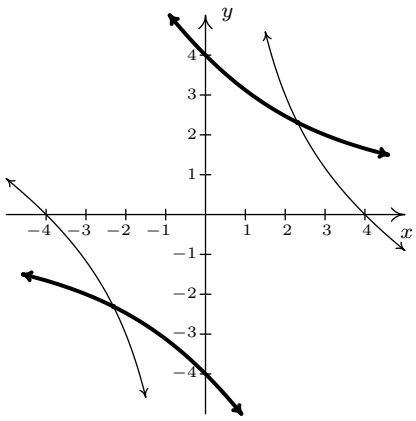
1. $\begin{cases} x^2 + 2xy - 16 &= 0 \\ y^2 + 2xy - 16 &= 0 \end{cases}$ 2. $\begin{cases} y + 4e^{2x} &= 1 \\ y^2 + 2e^x &= 1 \end{cases}$ 3. $\begin{cases} z(x-2) &= x \\ yz &= y \\ (x-2)^2 + y^2 &= 1 \end{cases}$

Solution.

1. At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome $xy$ term, and the constant term as well, by elimination and doing so we get a more tractable relationship between $x$ and $y$

$$\begin{cases} (E1) & x^2 + 2xy - 16 &= 0 \\ (E2) & y^2 + 2xy - 16 &= 0 \end{cases} \xrightarrow[-E1 + E2]{\text{Replace } E2 \text{ with}} \begin{cases} (E1) & x^2 + 2xy - 16 &= 0 \\ (E2) & y^2 - x^2 &= 0 \end{cases}$$

We get $y^2 - x^2 = 0$ or $y = \pm x$. Substituting $y = x$ into $E1$ we get $x^2 + 2x^2 - 16 = 0$ so that $x^2 = \frac{16}{3}$ or $x = \pm\frac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into $E1$, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$ which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. We leave it to the reader to show that both points satisfy both equations and now turn to verifying our solution graphically. We begin by solving $x^2+2xy-16 = 0$ for $y$ to obtain $y = \frac{16-x^2}{2x}$. This function is easily graphed using the techniques of Section 5.2. Solving the second equation, $y^2 + 2xy - 16 = 0$, for $y$, however, is more complicated. We use the quadratic formula to obtain $y = -x \pm \sqrt{x^2 + 16}$ which would require the use of Calculus or a calculator to graph. Believe it or not, we don't need either because the equation $y^2+2xy-16 = 0$ can be obtained from the equation $x^2 + 2xy - 16 = 0$ by interchanging $y$ and $x$. Thinking back to Section 6.2, this means we can obtain the graph of $y^2 + 2xy - 16 = 0$ by reflecting the graph of $x^2 + 2xy - 16 = 0$ across the line $y = x$. Doing so confirms that the two graphs intersect twice: once in Quadrant I, and once in Quadrant III as required.




The graphs of $x^2 + 2xy - 16 = 0$ and $\boldsymbol{y^2 + 2xy - 16 = 0}$

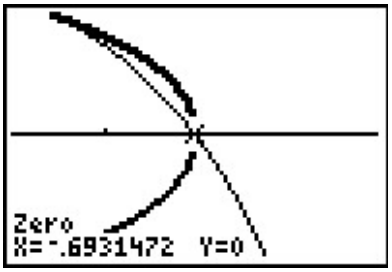
2. Unlike the previous problem, there seems to be no avoiding substitution and a bit of algebraic unpleasantness. Solving $y + 4e^{2x} = 1$ for $y$, we get $y = 1 - 4e^{2x}$ which, when substituted into the second equation, yields $\left(1 - 4e^{2x}\right)^2 + 2e^x = 1$. After expanding and gathering like terms, we get $16e^{4x} - 8e^{2x} + 2e^x = 0$. Factoring gives us $2e^x\left(8e^{3x} - 4e^x + 1\right) = 0$, and since $2e^x \neq 0$ for any real $x$, we are left with solving $8e^{3x} - 4e^x + 1 = 0$. We have three terms, and even though this is not a 'quadratic in disguise', we can benefit from the substitution $u = e^x$. The equation becomes $8u^3 - 4u + 1 = 0$. We find $u = \frac{1}{2}$ is a zero and use synthetic division to factor the left hand side as $\left(u - \frac{1}{2}\right)\left(8u^2 + 4u - 2\right)$. We use the quadratic formula to solve $8u^2 + 4u - 2 = 0$ and find $u = \frac{-1 \pm \sqrt{5}}{4}$. Since $u = e^x$, we now must solve $e^x = \frac{1}{2}$ and $e^x = \frac{-1 \pm \sqrt{5}}{4}$. From $e^x = \frac{1}{2}$, we get $x = \ln\left(\frac{1}{2}\right) = -\ln(2)$. As for $e^x = \frac{-1 \pm \sqrt{5}}{4}$, we first note that $\frac{-1-\sqrt{5}}{4} < 0$, so $e^x = \frac{-1-\sqrt{5}}{4}$ has no real solutions. We are left with $e^x = \frac{-1+\sqrt{5}}{4}$, so that $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$. We now return to $y = 1 - 4e^{2x}$ to find the accompanying $y$ values for each of our solutions for $x$. For $x = -\ln(2)$, we get

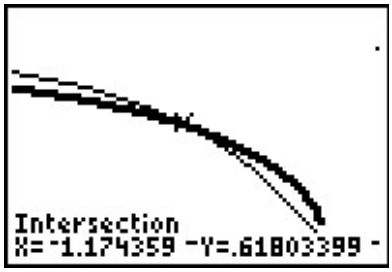
$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{-2\ln(2)} \\ &= 1 - 4e^{\ln(1/4)} \\ &= 1 - 4\left(\tfrac{1}{4}\right) \\ &= 0 \end{aligned}$$

For $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$, we have

$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{2\ln\left(\frac{-1+\sqrt{5}}{4}\right)} \\ &= 1 - 4e^{\ln\left(\frac{-1+\sqrt{5}}{4}\right)^2} \\ &= 1 - 4\left(\frac{-1+\sqrt{5}}{4}\right)^2 \\ &= 1 - 4\left(\frac{3-\sqrt{5}}{8}\right) \\ &= \frac{-1+\sqrt{5}}{2} \end{aligned}$$

We get two solutions, $\left\{(0, -\ln(2)), \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)\right\}$. It is a good review of the properties of logarithms to verify both solutions, so we leave that to the reader. We are able to sketch $y = 1 - 4e^{2x}$ using transformations, but the second equation is more difficult and we resort to the calculator. We note that to graph $y^2 + 2e^x = 1$, we need to graph both the positive and negative roots, $y = \pm\sqrt{1 - 2e^x}$. After some careful zooming,[2] we confirm our solutions.




The graphs of $y = 1 - 4e^{2x}$ and $\boldsymbol{y = \pm\sqrt{1 - 2e^x}}$.

3. Our last system involves three variables and gives some insight on how to keep such systems organized. Labeling the equations as before, we have

$$\begin{cases} E1 & z(x-2) &= x \\ E2 & yz &= y \\ E3 & (x-2)^2 + y^2 &= 1 \end{cases}$$

The easiest equation to start with appears to be $E2$. While it may be tempting to divide both sides of $E2$ by $y$, we caution against this practice because it presupposes $y \neq 0$. Instead,

[2] The calculator has trouble confirming the solution $(-\ln(2), 0)$ due to its issues in graphing square root functions. If we mentally connect the two branches of the thicker curve, we see the intersection.

we take $E2$ and rewrite it as $yz - y = 0$ so $y(z-1) = 0$. From this, we get two cases: $y = 0$ or $z = 1$. We take each case in turn.

CASE 1: $y = 0$. Substituting $y = 0$ into $E1$ and $E3$, we get

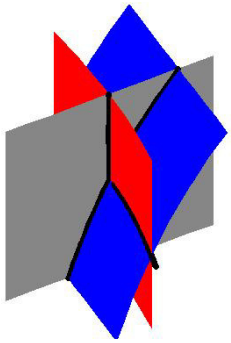
$$\begin{cases} E1 & z(x-2) &=& x \\ E3 & (x-2)^2 &=& 1 \end{cases}$$

Solving $E3$ for $x$ gives $x = 1$ or $x = 3$. Substituting these values into $E1$ gives $z = -1$ when $x = 1$ and $z = 3$ when $x = 3$. We obtain two solutions, $(1, 0, -1)$ and $(3, 0, 3)$.

CASE 2: $z = 1$. Substituting $z = 1$ into $E1$ and $E3$ gives us

$$\begin{cases} E1 & (1)(x-2) &=& x \\ E3 & (1-2)^2 + y^2 &=& 1 \end{cases}$$

Equation $E1$ gives us $x - 2 = x$ or $-2 = 0$, which is a contradiction. This means we have no solution to the system in this case, even though $E3$ is solvable and gives $y = 0$. Hence, our final answer is $\{(1, 0, -1), (3, 0, 3)\}$. These points are easy enough to check algebraically in our three original equations, so that is left to the reader. As for verifying these solutions graphically, they require plotting surfaces in three dimensions and looking for intersection points. While this is beyond the scope of this book, we provide a snapshot of the graphs of our three equations near one of the solution points, $(1, 0, -1)$.

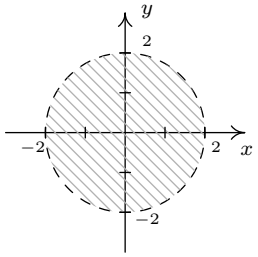


□

Example 8.4.2 showcases some of the ingenuity and tenacity mentioned at the beginning of the section. Sometimes you just have to look at a system the right way to find the most efficient method to solve it. Sometimes you just have to try something.

We close this section discussing how non-linear inequalities can be used to describe regions in the plane which we first introduced in Section 3.4. Before we embark on some examples, a little motivation is in order. Suppose we wish to solve $x^2 < 4 - y^2$. If we mimic the algorithms for solving nonlinear inequalities in one variable, we would gather all of the terms on one side and leave a 0

on the other to obtain $x^2 + y^2 - 4 < 0$. Then we would find the zeros of the left hand side, that is, where is $x^2 + y^2 - 4 = 0$, or $x^2 + y^2 = 4$. Instead of obtaining a few *numbers* which divide the real number *line* into *intervals*, we get an equation of a *curve*, in this case, a circle, which divides the *plane* into two *regions* - the 'inside' and 'outside' of the circle - with the circle itself as the boundary between the two. Just like we used test *values* to determine whether or not an interval belongs to the solution of the inequality, we use test *points* in the each of the regions to see which of these belong to our solution set.[3] We choose $(0, 0)$ to represent the region inside the circle and $(0, 3)$ to represent the points outside of the circle. When we substitute $(0, 0)$ into $x^2 + y^2 - 4 < 0$, we get $-4 < 4$ which is true. This means $(0, 0)$ and all the other points inside the circle are part of the solution. On the other hand, when we substitute $(0, 3)$ into the same inequality, we get $5 < 0$ which is false. This means $(0, 3)$ along with all other points outside the circle are not part of the solution. What about points on the circle itself? Choosing a point on the circle, say $(0, 2)$, we get $0 < 0$, which means the circle itself does not satisfy the inequality.[4] As a result, we leave the circle dashed in the final diagram.




The solution to $x^2 < 4 - y^2$

We put this technique to good use in the following example.

Example 8.4.3. Sketch the solution to the following nonlinear inequalities in the plane.

1. $y^2 - 4 \leq x < y + 2$
2. $\begin{cases} x^2 + y^2 & \geq & 4 \\ x^2 - 2x + y^2 - 2y & \leq & 0 \end{cases}$

Solution.

1. The inequality $y^2 - 4 \leq x < y + 2$ is a compound inequality. It translates as $y^2 - 4 \leq x$ and $x < y + 2$. As usual, we solve each inequality and take the set theoretic intersection to determine the region which satisfies both inequalities. To solve $y^2 - 4 \leq x$, we write $y^2 - x - 4 \leq 0$. The curve $y^2 - x - 4 = 0$ describes a parabola since exactly one of the variables is squared. Rewriting this in standard form, we get $y^2 = x + 4$ and we see that the vertex is $(-4, 0)$ and the parabola opens to the right. Using the test points $(-5, 0)$ and $(0, 0)$,

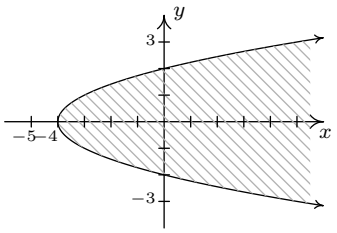
[3] The theory behind why all this works is, surprisingly, the same theory which guarantees that sign diagrams work the way they do - continuity and the Intermediate Value Theorem - but in this case, applied to functions of more than one variable.

[4] Another way to see this is that points on the circle satisfy $x^2 + y^2 - 4 = 0$, so they do not satisfy $x^2 + y^2 - 4 < 0$.

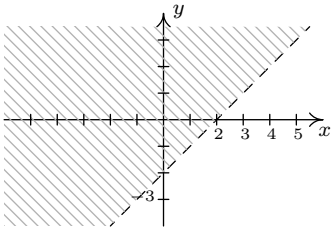
we find that the solution to the inequality includes the region to the right of, or 'inside', the parabola. The points on the parabola itself are also part of the solution, since the vertex $(-4, 0)$ satisfies the inequality. We now turn our attention to $x < y+2$. Proceeding as before, we write $x - y - 2 < 0$ and focus our attention on $x - y - 2 = 0$, which is the line $y = x - 2$. Using the test points $(0, 0)$ and $(0, -4)$, we find points in the region above the line $y = x - 2$ satisfy the inequality. The points on the line $y = x - 2$ do not satisfy the inequality, since the $y$-intercept $(0, -2)$ does not. We see that these two regions do overlap, and to make the graph more precise, we seek the intersection of these two curves. That is, we need to solve the system of nonlinear equations

$$\begin{cases} (E1) & y^2 = x + 4 \\ (E2) & y = x - 2 \end{cases}$$

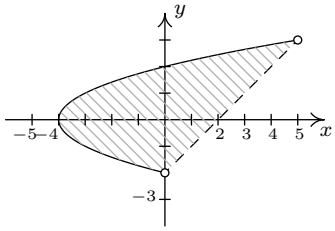
Solving $E1$ for $x$, we get $x = y^2 - 4$. Substituting this into $E2$ gives $y = y^2 - 4 - 2$, or $y^2 - y - 6 = 0$. We find $y = -2$ and $y = 3$ and since $x = y^2 - 4$, we get that the graphs intersect at $(0, -2)$ and $(5, 3)$. Putting all of this together, we get our final answer below.




$y^2 - 4 \leq x$




$x < y + 2$

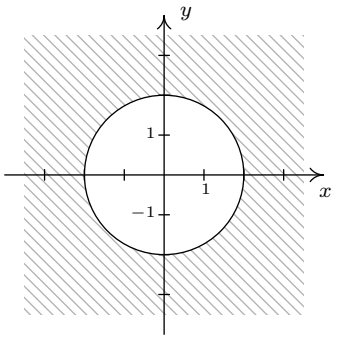



$y^2 - 4 \leq x < y + 2$

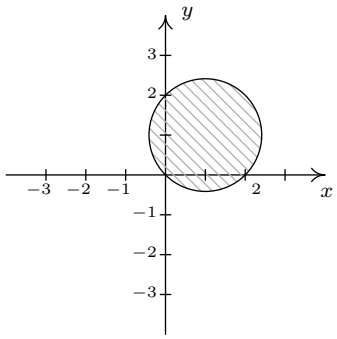
2. To solve this system of inequalities, we need to find all of the points $(x, y)$ which satisfy both inequalities. To do this, we solve each inequality separately and take the set theoretic intersection of the solution sets. We begin with the inequality $x^2 + y^2 \geq 4$ which we rewrite as $x^2 + y^2 - 4 \geq 0$. The points which satisfy $x^2 + y^2 - 4 = 0$ form our friendly circle $x^2 + y^2 = 4$. Using test points $(0, 0)$ and $(0, 3)$ we find that our solution comprises the region outside the circle. As far as the circle itself, the point $(0, 2)$ satisfies the inequality, so the circle itself is part of the solution set. Moving to the inequality $x^2 - 2x + y^2 - 2y \leq 0$, we start with $x^2 - 2x + y^2 - 2y = 0$. Completing the squares, we obtain $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle centered at $(1, 1)$ with a radius of $\sqrt{2}$. Choosing $(1, 1)$ to represent the inside of the circle, $(1, 3)$ as a point outside of the circle and $(0, 0)$ as a point on the circle, we find that the solution to the inequality is the inside of the circle, including the circle itself. Our final answer, then, consists of the points on or outside of the circle $x^2 + y^2 = 4$ which lie on or inside the circle $(x - 1)^2 + (y - 1)^2 = 2$. To produce the most accurate graph, we need to find where these circles intersect. To that end, we solve the system

$$\begin{cases} (E1) & x^2 + y^2 &= 4 \\ (E2) & x^2 - 2x + y^2 - 2y &= 0 \end{cases}$$

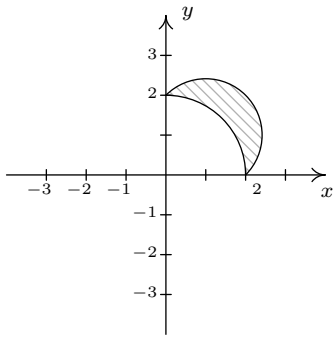
We can eliminate both the $x^2$ and $y^2$ by replacing $E2$ with $-E1 + E2$. Doing so produces $-2x - 2y = -4$. Solving this for $y$, we get $y = 2 - x$. Substituting this into $E1$ gives $x^2 + (2 - x)^2 = 4$ which simplifies to $x^2 + 4 - 4x + x^2 = 4$ or $2x^2 - 4x = 0$. Factoring yields $2x(x - 2)$ which gives $x = 0$ or $x = 2$. Substituting these values into $y = 2 - x$ gives the points $(0, 2)$ and $(2, 0)$. The intermediate graphs and final solution are below.




$x^2 + y^2 \geq 4$




$x^2 - 2x + y^2 - 2y \leq 0$




Solution to the system.

□

### 8.4.1 EXERCISES

1. Solve the following systems of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

   (a) $\begin{cases} x^2 - y &= 4 \\ x^2 + y^2 &= 4 \end{cases}$ (c) $\begin{cases} x^2 + y^2 &= 16 \\ 16x^2 + 4y^2 &= 64 \end{cases}$ (e) $\begin{cases} x^2 + y^2 &= 16 \\ \frac{1}{9}y^2 - \frac{1}{16}x^2 &= 1 \end{cases}$

   (b) $\begin{cases} x^2 + y^2 &= 4 \\ x^2 - y &= 5 \end{cases}$ (d) $\begin{cases} x^2 + y^2 &= 16 \\ 9x^2 - 16y^2 &= 144 \end{cases}$ (f) $\begin{cases} x^2 + y^2 &= 16 \\ x - y &= 2 \end{cases}$

2. Solve the following systems of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

   (a) $\begin{cases} x^2 - y^2 &= 1 \\ x^2 + 4y^2 &= 4 \end{cases}$ (d) $\begin{cases} (x-2)^2 + y^2 &= 1 \\ x^2 + 4y^2 &= 4 \end{cases}$ (g) $\begin{cases} y &= x^3 + 8 \\ y &= 10x - x^2 \end{cases}$

   (b) $\begin{cases} \sqrt{x+1} - y &= 0 \\ x^2 + 4y^2 &= 4 \end{cases}$ (e) $\begin{cases} x^2 + y^2 &= 25 \\ y - x &= 1 \end{cases}$ (h) $\begin{cases} x^2 + y^2 &= 25 \\ 4x^2 - 9y &= 0 \\ 3y^2 - 16x &= 0 \end{cases}$

   (c) $\begin{cases} x + 2y^2 &= 2 \\ x^2 + 4y^2 &= 4 \end{cases}$ (f) $\begin{cases} x^2 + y^2 &= 25 \\ x^2 + (y-3)^2 &= 10 \end{cases}$

3. Consider the system of nonlinear equations below

   $$\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} &= 1 \\ \dfrac{3}{x} + \dfrac{2}{y} &= -1 \end{cases}$$

   If we let $u = \frac{1}{x}$ and $v = \frac{1}{y}$ then the system becomes

   $$\begin{cases} 4u + 3v &= 1 \\ 3u + 2v &= -1 \end{cases}$$

   This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u = -5$ and $v = 7$. Thus $x = \frac{1}{u} = -\frac{1}{5}$ and $y = \frac{1}{v} = \frac{1}{7}$.

   We say that the original system is **linear in form** because its equations are not linear but a few basic substitutions reveal a structure that we can treat like a system of linear equations. Each system given below is linear in form. Make the appropriate substitutions to help you solve for $x$ and $y$.

(a) $\begin{cases} 4x^3 + 3\sqrt{y} = 1 \\ 3x^3 + 2\sqrt{y} = -1 \end{cases}$ (b) $\begin{cases} 4e^x + 3e^{-y} = 1 \\ 3e^x + 2e^{-y} = -1 \end{cases}$ (c) $\begin{cases} 4\ln(x) + 3y^2 = 1 \\ 3\ln(x) + 2y^2 = -1 \end{cases}$

4. Solve the following system

$$\begin{cases} x^2 + \sqrt{y} + \log_2(z) = 6 \\ 3x^2 - 2\sqrt{y} + 2\log_2(z) = 5 \\ -4x^2 + \sqrt{y} - 3\log_2(z) = 11 \end{cases}$$

5. Sketch the solution to each system of nonlinear inequalities in the plane.

(a) $\begin{cases} x^2 - y^2 \leq 1 \\ x^2 + 4y^2 \geq 4 \end{cases}$

(b) $\begin{cases} x^2 + y^2 < 25 \\ x^2 + (y-3)^2 \geq 10 \end{cases}$

(c) $\begin{cases} (x-2)^2 + y^2 < 1 \\ x^2 + 4y^2 < 4 \end{cases}$

(d) $\begin{cases} y > 10x - x^2 \\ y < x^3 + 8 \end{cases}$

(e) $\begin{cases} x + 2y^2 > 2 \\ x^2 + 4y^2 \leq 4 \end{cases}$

(f) $\begin{cases} x^2 + y^2 \geq 25 \\ y - x \leq 1 \end{cases}$

6. Systems of nonlinear equations show up in third semester Calculus in the midst of some really cool problems. The system below came from a problem in which we were asked to find the dimensions of a rectangular box with a volume of 1000 cubic inches that has minimal surface area. The variables $x$, $y$ and $z$ are the dimensions of the box and $\lambda$ is called a Lagrange multiplier. With the help of your classmates, solve the system.[5]

$$\begin{cases} 2y + 2z = \lambda yz \\ 2x + 2z = \lambda xz \\ 2y + 2x = \lambda xy \\ xyz = 1000 \end{cases}$$

7. According to Theorem 10.5 in Section 10.1, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

   (a) Show that $p$ has no real zeros.

[5] If using $\lambda$ bothers you, change it to $w$ when you solve the system.

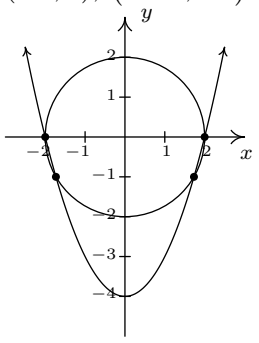
(b) Because $p$ has no real zeros, its factorization must be of the form $(x^2+ax+b)(x^2+cx+d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

(c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns $a$, $b$, $c$ and $d$. Write $p(x)$ in factored form.

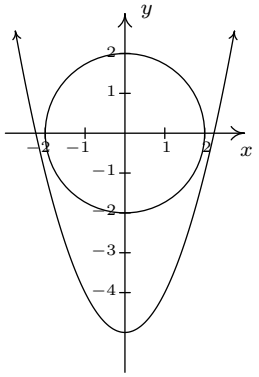
8. Factor $q(x) = x^4 + 6x^2 - 5x + 6$.

### 8.4.2 Answers

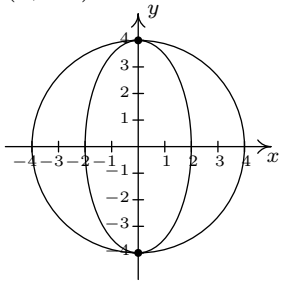
1. (a) $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$



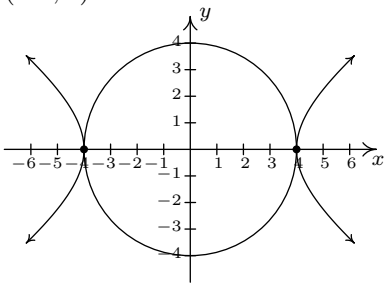
(b) No solution



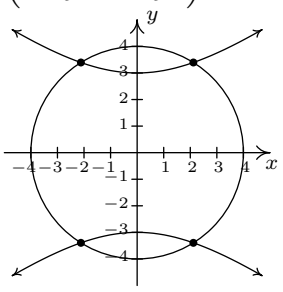
(c) $(0, \pm 4)$



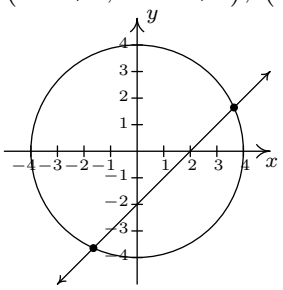
(d) $(\pm 4, 0)$



(e) $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$



(f) $\left(1+\sqrt{7}, -1+\sqrt{7}\right)$, $\left(1-\sqrt{7}, -1-\sqrt{7}\right)$

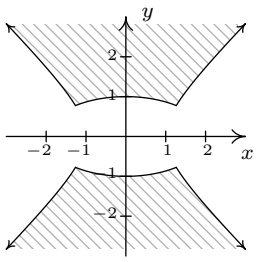


2. (a) $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$
   (b) $(0, 1)$
   (c) $(0, \pm 1)$, $(2, 0)$
   (d) $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{9}\right)$
   (e) $(3, 4)$, $(-4, -3)$
   (f) $(\pm 3, 4)$
   (g) $(-4, -56)$, $(1, 9)$, $(2, 16)$
   (h) $(3, 4)$

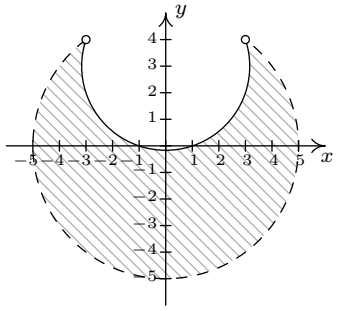
3. (a) $\left(-\sqrt[3]{5}, 49\right)$ (b) No solution (c) $\left(e^{-5}, \sqrt{7}\right)$

4. $(1, 4, 8),\ (-1, 4, 8)$

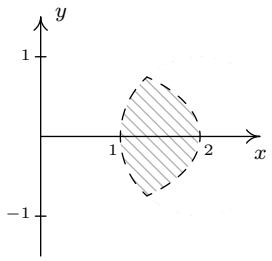
5. (a) $\begin{cases} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{cases}$



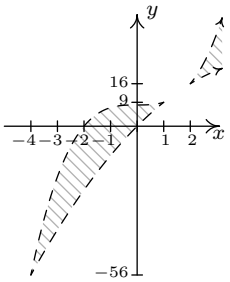

(b) $\begin{cases} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{cases}$



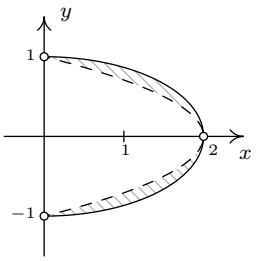

(c) $\begin{cases} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{cases}$



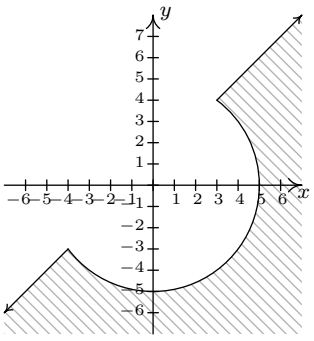
(d) $\begin{cases} y > 10x - x^2 \\ y < x^3 + 8 \end{cases}$




(e) $\begin{cases} x + 2y^2 > 2 \\ x^2 + 4y^2 \leq 4 \end{cases}$




(f) $\begin{cases} x^2 + y^2 \geq 25 \\ y - x \leq 1 \end{cases}$




6. $x = 10,\ y = 10,\ z = 10, \lambda = \frac{2}{5}$

7. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

8. $x^4 + 6x^2 - 5x + 6 = (x^2 - x + 1)(x^2 + x + 6)$

# Chapter 9

# Sequences and Series

## 9.1 Sequences

When we first introduced a function as a special type of relation in Section 2.1, we did not put any restrictions on the domain of the function. All we said was that the set of $x$-coordinates of the points in the function $F$ is called the domain, and it turns out that any subset of the real numbers, regardless of how weird that subset may be, can be the domain of a function. As our exploration of functions continued beyond Section 2.1, we saw fewer and fewer functions with 'weird' domains. It is worth your time to go back through the text to see that the domains of the polynomial, rational, exponential, logarithmic and algebraic functions discussed thus far have fairly predictable domains which almost always consist of just a collection of intervals on the real line. This may lead some readers to believe that the only important functions in a College Algebra text have domains which consist of intervals and everything else was just introductory nonsense. In this section, we introduce **sequences** which are an important class of functions whose domains are the set of natural numbers.[1] Before we get to far ahead of ourselves, let's look at what the term 'sequence' means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \ldots \tag{1}$$

As usual, the periods of ellipsis, ..., indicate that the proposed pattern continues forever. Each of the numbers in the list is called a **term**, and we call $\frac{1}{2}$ the 'first term', $-\frac{3}{4}$ the 'second term', $\frac{9}{8}$ the 'third term' and so forth. In numbering them this way, we are setting up a function, which we'll call $a$ per tradition, between the natural numbers and the terms in the sequence.

[1] Recall that this is the set $\{1, 2, 3, \ldots\}$.

| $n$ | $a(n)$ |
|---|---|
| 1 | $\frac{1}{2}$ |
| 2 | $-\frac{3}{4}$ |
| 3 | $\frac{9}{8}$ |
| 4 | $-\frac{27}{16}$ |
| $\vdots$ | $\vdots$ |

In other words, $a(n)$ is the $n^{\text{th}}$ term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

DEFINITION 9.1. A **sequence** is a function $a$ whose domain is the natural numbers. The value $a(n)$ is often written as $a_n$ and is called the $\boldsymbol{n^{\text{th}}}$ **term** of the sequence. The sequence itself is usually denoted $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 9.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$ and so forth. Now suppose we wanted to know $a_{117}$, that is, the $117^{\text{th}}$ term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula for $a_n$. Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

EXAMPLE 9.1.1. Write the first four terms of the following sequences.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\left\{\dfrac{1+(-1)^j}{j}\right\}_{j=2}^{\infty}$
5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$
6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$

SOLUTION.

1. Since we are given $n \geq 1$, the first four terms of the sequence are $a_1$, $a_2$, $a_3$ and $a_4$. Since the notation $a_1$ means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of $n$ in the formula for $a_n$ with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

2. For this sequence we have $k \geq 0$, so the first four terms are $b_0$, $b_1$, $b_2$ and $b_3$. Proceeding as before, replacing in this case the variable $k$ with the appropriate whole number, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. (This sequence is called an **alternating** sequence since the signs alternate between $+$ and $-$. The reader is encouraged to think what component of the formula is producing this effect.)

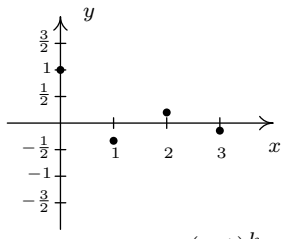
3. From $\{2n-1\}_{n=1}^{\infty}$, we have that $a_n = 2n-1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$. (The first four terms are the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.)

4. Proceeding as in the previous problem, we set $a_j = \frac{1+(-1)^j}{j}$, $j \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

5. To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

6. As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. □

Some remarks about Example 9.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 9.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of $k$ in the domain, $k = 0, 1, 2, \ldots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers. If you feel a sense of nostalgia, you should see Section 1.2.




Graphing $y = b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 9.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other

words, to satisfy Definition 9.1, we need to shift the variable $k$ so it starts at $k = 1$ instead of $k = 0$. To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section 2.5, we can accomplish this by replacing $k$ with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$. Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition 9.1, while the latter does not. Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 9.1 to include any function which generates a list of numbers which can be then be matched up with the natural numbers.[2] Finally, we wish to note the sequences in parts 5 and 6 are examples of sequences described **recursively**. In each instance, an initial value of the sequence is given which is then followed by a **recursion equation** – a formula which enables us to use known terms of the sequence to determine other terms. The terms of the sequence in part 6 are given a special name: $f_n$ is called ***n*-factorial**. We will study factorials in greater detail in Section **??**. The world famous Fibonacci Numbers are defined recursively and are explored in the exercises. While none of the sequences worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

DEFINITION 9.2. **Arithmetic and Geometric Sequences:** Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence[a]

- If there is a number $d$ so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number $d$ is called the **common difference**.
- If there is a number $r$ so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **geometric sequence**. The number $r$ is called the **common ratio**.

[a]Note that we have adjusted for the fact that not all 'sequences' begin at $n = 1$.

Both arithmetic and geometric sequences are defined in terms of recursion equations. In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number $d$. The name 'common difference' comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number $r$. If $r \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$, hence the name 'common ratio.' Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.[3]

EXAMPLE 9.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference $d$; if geometric, find the common ratio $r$.

[2]Math fans will delight to know we are basically talking about the 'countably infinite' subsets of the real number line when we do this.

[3]Sequences which are both arithmetic and geometric are discussed in the Exercises.

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$
2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$
3. $\{2n-1\}_{n=1}^{\infty}$
4. $\dfrac{1}{2}, -\dfrac{3}{4}, \dfrac{9}{8}, -\dfrac{27}{16}, \ldots$

SOLUTION. A good rule of thumb to keep in mind when working with sequences is "When in doubt, write it out!" Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 9.1.1, we know that the first four terms of this sequence are $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' and we have established that the sequence is *not* arithmetic. To investigate whether or not it is geometric, we compute the ratios of successive terms. The first three ratios
$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$
suggest that the sequence is geometric. To prove it, we must show that $\frac{a_{n+1}}{a_n} = r$ for all $n$.
$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{(n+1)-1}}{3^{n+1}}}{\frac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$
This sequence is geometric with common ratio $r = \frac{5}{3}$.

2. Again, we have Example 9.1.1 to thank for providing the first four terms of this sequence: $1, -\frac{1}{3}, \frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no 'common ratio,' we conclude the sequence is not geometric, either.

3. As we saw in Example 9.1.1, the sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To verify this, we find
$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$
This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

4. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we assume that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. □

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

EQUATION 9.1. **Formulas for Arithmetic and Geometric Sequences:**

- An arithmetic sequence with first term $a$ and common difference $d$ is given by
$$a_n = a + (n-1)d, \quad n \geq 1$$
- A geometric sequence with first term $a$ and common ratio $r \neq 0$ is given by
$$a_n = ar^{n-1}, \quad n \geq 1$$

While the formal proofs of the formulas in Equation 9.1 require the techniques set forth in Section **??**, we attempt to motivate them here. According to Definition 9.2, given an arithmetic sequence with first term $a$ and common difference $d$, the way we get from one term to the next is by adding $d$. Hence, the terms of the sequence are: $a$, $a+d$, $a+2d$, $a+3d$, .... We see that to reach the $n$th term, we add $d$ to $a$ exactly $(n-1)$ times, which is what the formula says. The derivation of the formula for geometric series follows similarly. Here, we start with $a$ and go from one term to the next by multiplying by $r$. We get $a, ar, ar^2, ar^3$ and so forth. The $n$th term results from multiplying $a$ by $r$ exactly $(n-1)$ times. We note here that the reason $r = 0$ is excluded from Equation 9.1 is to avoid an instance of $0^0$ which is an indeterminant form.[4] With Equation 9.1 in place, we finally have the tools required to find an explicit formula for the $n$th term of the sequence given in (1). We know from Example 9.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 9.1 we get $a_n = ar^{n-1} = \frac{1}{2}\left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of $n$ and seeing that the formula generates the sequence given in (1). We leave this to the reader. Our next example gives us more practice finding patterns.

EXAMPLE 9.1.3. Find an explicit formula for the $n^{\text{th}}$ term of the following sequences.

1. $0.9, 0.09, 0.009, 0.0009, \ldots$
2. $\frac{2}{5}, 2, -\frac{2}{3}, -\frac{2}{7}, \ldots$
3. $1, -\frac{2}{7}, \frac{4}{13}, -\frac{8}{19}, \ldots$

SOLUTION.

[4] See the footnotes on page 217 in Section 4.1 and page 333 of Section 7.1.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10}\left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.

2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions. We have $\frac{2}{5}, \frac{2}{1}, -\frac{2}{3}, -\frac{2}{7}, \ldots$. If we associate the negative '$-$' of the last two terms with the denominators we get $\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, \frac{2}{-7}, \ldots$. This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{d_n}$ where $d_n$ is the sequence of denominators. Looking at the denominators $5, 1, -3, -7, \ldots$, we find that they go from one term to the next by subtracting 4 which is the same as adding $-4$. This means we have an arithmetic sequence on our hands. Using Equation 9.1 with $a = 5$ and $d = -4$, we get the $n$th denominator by the formula $d_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting $c_n$ and $d_n$ denote the sequence of numerators and denominators, respectively, we have $a_n = \frac{c_n}{d_n}$. After some experimentation,[5] we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields $\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$. The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $c_n = (-2)^{n-1}$, for $n \geq 1$. The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $d_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$. We obtain our formula for $a_n = \frac{c_n}{d_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. □

While the last problem in Example 9.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \ldots$, we would be hard-pressed to find a formula for $a_n$ if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the $n$th term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns. The pattern $1, 4, 9, 16, \ldots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \ldots$ as the sequence $1 + 1, 4 + 1, 9 + 1, 16 + 1, \ldots$, so that $a_n = n^2 + 1$, $n \geq 1$. Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.[6] Just because your answer doesn't match ours doesn't mean

[5] Here we take 'experimentation' to mean a frustrating guess-and-check session.

[6] For more on this, see When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test by Professor Don White of Kent State University.

it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

We would be remiss to close this section without mention of the utility of sequences in everyday life. Indeed, sequences play a major role in the Mathematics of Finance, as we have already seen with Equation 7.2 in Section 7.5. Recall that if we invest $P$ dollars at an annual percentage rate $r$ and compound the interest $n$ times per year, the formula for $A_k$, the amount in the account after $k$ compounding periods, is $A_k = P\left(1+\frac{r}{n}\right)^k = \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right)^{k-1}$, $k \geq 1$. We now spot this as a geometric sequence with first term $P\left(1+\frac{r}{n}\right)$ and common ratio $\left(1+\frac{r}{n}\right)$. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an **Investment Retirement Account** and will be discussed in the next section once we have developed more mathematical machinery.

### 9.1.1 EXERCISES

1. Write out the four terms of the following sequences.

   (a) $a_n = 2^n - 1$, $n \geq 0$
   (b) $d_j = (-1)^{\frac{j(j+1)}{2}}$, $j \geq 1$
   (c) $\{5k-2\}_{k=1}^{\infty}$
   (d) $\left\{\frac{n^2+1}{n+1}\right\}_{n=0}^{\infty}$
   (e) $\left\{\frac{x^n}{n^2}\right\}_{n=1}^{\infty}$
   (f) $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$
   (g) $a_1 = 3$, $a_{n+1} = a_n - 1$, $n \geq 1$
   (h) $d_0 = 12$, $d_m = \frac{d_{m-1}}{100}$, $m \geq 1$
   (i) $b_1 = 2$, $b_{k+1} = 3b_k + 1$, $k \geq 1$
   (j) $c_0 = -2$, $c_j = \frac{c_{j-1}}{(j+1)(j+2)}$, $m \geq 1$
   (k) $a_1 = 117$, $a_{n+1} = \frac{1}{a_n}$, $n \geq 1$
   (l) $s_0 = 1$, $s_{n+1} = x^{n+1} + s_n$, $n \geq 0$
   (m) $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ (This is the famous Fibonacci Sequence )

2. Find an explicit formula for the $n^{\text{th}}$ term of the following sequences. Use the formulas in Equation 9.1 as needed.

   (a) $3, 5, 7, 9, \ldots$
   (b) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$
   (c) $1, \frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \ldots$
   (d) $1, \frac{2}{3}, \frac{1}{3}, \frac{4}{27}, \ldots$
   (e) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$
   (f) $x, -\frac{x^3}{3}, \frac{x^5}{5}, -\frac{x^7}{7}, \ldots$
   (g) $0.9, 0.99, 0.999, 0.9999, \ldots$
   (h) $27, 64, 125, 216, \ldots$
   (i) $1, 0, 1, 0, \ldots$

3. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all $n$.)

4. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.

5. Thomas Robert Malthus is credited with saying, "The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second." (See this webpage for more information.) Discuss this quote with your classmates from a sequences point of view.
6. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the $30^{\text{th}}$ day? What is the total dollar value of the gift you have received?
7. Research the terms 'arithmetic mean' and 'geometric mean.' With the help of your classmates, show that a given term of a arithmetic sequence $a_k$, $k \geq 2$ is the arithmetic mean of the term immediately preceding, $a_{k-1}$ it and immediately following it, $a_{k+1}$. State and prove an analogous result for geometric sequences.
8. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the $n^{\text{th}}$ term of the sequence $i, -1, -i, 1, i, \ldots$. List out the first four terms of the matrix sequences we discussed in Exercise **??** in Section **??**.

### 9.1.2 Answers

1. (a) $0, 1, 3, 7$
   (b) $-1, -1, 1, 1$
   (c) $3, 8, 13, 18$
   (d) $1, 1, \frac{5}{3}, \frac{5}{2}$
   (e) $x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}$
   (f) $0, \ln(2), \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$
   (g) $3, 2, 1, 0$
   (h) $12, 0.12, 0.0012, 0.000012$
   (i) $2, 7, 22, 67$
   (j) $-2, -\frac{1}{3}, -\frac{1}{36}, -\frac{1}{720}$
   (k) $117, \frac{1}{117}, 117, \frac{1}{117}$
   (l) $1, x+1, x^2+x+1, x^3+x^2+x+1$
   (m) $1, 1, 2, 3$
2. (a) $a_n = 1 + 2n,\ n \geq 1$
   (b) $a_n = \left(-\frac{1}{2}\right)^{n-1},\ n \geq 1$
   (c) $a_n = \frac{2^{n-1}}{2n-1},\ n \geq 1$
   (d) $a_n = \frac{n}{3^{n-1}},\ n \geq 1$
   (e) $a_n = \frac{1}{n^2},\ n \geq 1$
   (f) $\frac{(-1)^{n-1}x^{2n-1}}{2n-1},\ n \geq 1$
   (g) $a_n = \frac{10^n - 1}{10^n},\ n \geq 1$
   (h) $a_n = (n+2)^3,\ n \geq 1$
   (i) $a_n = \frac{1+(-1)^{n-1}}{2},\ n \geq 1$

## 9.2 Series and Summation Notation

In the previous section, we introduced sequences and now we shall present notation and theorems concerning the sum of terms of a sequence. We begin with a definition, which, while intimidating, is meant to make our lives easier.

DEFINITION 9.3. **Summation Notation:** Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers $m$ and $p$ satisfying $k \leq m \leq p$, the summation from $m$ to $p$ of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable $n$ is called the **index of summation**. The number $m$ is called the **lower limit of summation** while the number $p$ is called the **upper limit of summation**.

In English, Definition 9.3 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from $a_m$ through $a_p$. The symbol $\Sigma$ is the capital Greek letter sigma and is shorthand for 'sum'. The index of summation tells us which term to start with and which term to end with. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write the sum $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned} \sum_{n=3}^{6}(2n-1) &= (2(3)-1) + (2(4)-1) + (2(5)-1) + (2(6)-1) \\ &= 5 + 7 + 9 + 11 \\ &= 32 \end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers $a_k$, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition 4.1. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the $i$th row $R_i$ of a matrix $A = [a_{ij}]_{m \times n}$ and the $j^{\text{th}}$ column $C_j$ of a matrix $B = [b_{ij}]_{n \times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition **??**. Our next example gives us practice with this new notation.

Example 9.2.1.

1. Find the following sums.

   (a) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (b) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

   (a) $1+3+5+\ldots+117$

   (b) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{1}{117}$

   (c) $0.9+0.09+0.009+\ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9$

Solution.

1. (a) We substitute $k=1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k=4$.

$$\begin{aligned}\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1}+\frac{13}{100^2}+\frac{13}{100^3}+\frac{13}{100^4}\\ &= 0.13+0.0013+0.000013+0.00000013\\ &= 0.13131313\end{aligned}$$

   (b) We proceed as in part (a), replacing the index $n$, but *not* the variable $x$, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1+\frac{(-1)^{2+1}}{2}(x-1)^2+\frac{(-1)^{3+1}}{3}(x-1)^3\\ &\quad+\frac{(-1)^{1+4}}{4}(x-1)^4+\frac{(-1)^{1+5}}{5}(x-1)^5\\ &= (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 9.1.

   (a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a=1$ and common difference $d=2$. We get a formula for the $n$th term of the sequence using Equation 9.1 to get $a_n = 1+(n-1)2 = 2n-1$, $n \geq 1$. At this stage, we have the formula

for the terms, namely $2n-1$, and the lower limit of the summation, $n=1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of $n$ produces the term 117. Setting $a_n = 117$, we get $2n-1 = 117$ or $n = 59$. Our final answer is

$$1+3+5+\ldots+117 = \sum_{n=1}^{59}(2n-1)$$

(b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

$$\frac{1}{1}+\frac{-1}{2}+\frac{1}{3}+\frac{-1}{4}+\ldots+\frac{1}{117}$$

The numerators, 1, $-1$, etc. can be described by the geometric sequence[1] $c_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $d_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n=1$ and $n=117$, respectively. Thus

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{1}{117} = \sum_{n=1}^{117}\frac{(-1)^{n-1}}{n}$$

(c) Thanks to Example 9.1.3, we know that one formula for the $n^{\text{th}}$ term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation. To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need a denominator of $10^n$. Hence, $n$ is the upper limit of summation. Since $n$ is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose $k$ and get

$$0.9+0.09+0.009+\ldots 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^{n}\frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation. While we shall not have much need of these properties in Algebra, they do play a great role in Calculus. Moreover, there is much to be learned by thinking about why the properties hold. We invite the reader to prove these results. To get started, remember, "When in doubt, write it out!"

[1] This is indeed a geometric sequence with first term $a=1$ and common ratio $r=-1$.
[2] It is an arithmetic sequence with first term $a=1$ and common difference $d=1$.
[3] To see why, try writing the summation using '$n$' as the index.

Theorem 9.1. **Properties of Summation Notation:** Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- $\displaystyle\sum_{n=m}^{p}(a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$
- $\displaystyle\sum_{n=m}^{p} c\,a_n = c\sum_{n=m}^{p} a_n$, for any real number $c$.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$, for any natural number $m \le j < j+1 \le p$.
- $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any whole number $r$.

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \ge 1$, we let $S$ denote the sum of the first $n$ terms. To derive a formula for $S$, we write it out in two different ways

$$\begin{array}{rcccccccccc} S &=& a &+& (a+d) &+& \ldots &+& (a+(n-2)d) &+& (a+(n-1)d) \\ S &=& (a+(n-1)d) &+& (a+(n-2)d) &+& \ldots &+& (a+d) &+& a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a+(n-1)d) + (2a+(n-1)d) + \ldots + (2a+(n-1)d) + (2a+(n-1)d)$$

The right hand side of this equation contains $n$ terms, all of which are equal to $(2a+(n-1)d)$ so we get $2S = n(2a+(n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

$$S = \frac{n}{2}(2a+(n-1)d)$$

If we rewrite the quantity $2a+(n-1)d$ as $a + (a+(n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms $n$ and the *average* of the first and $n^{\text{th}}$ terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \ge 1$, and let $S$ once again denote the sum of the first $n$ terms. Comparing $S$ and $rS$, we get

$$\begin{array}{rcccccccccccc} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except $a$ and $ar^n$ to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a\,(1-r^n)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute $a$ through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.

EQUATION 9.2. **Sums of Arithmetic and Geometric Sequences:**

- The sum $S$ of the first $n$ terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum $S$ of the first $n$ terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \displaystyle\sum_{k=1}^{n} a_k = \frac{a_1 - a_{n+1}}{1-r} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$.
2. $S = \displaystyle\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 9.2, formal proofs require the machinery in Section **??**. An application of the arithmetic sum formula which proves useful in Calculus results in formula for the sum of the first $n$ natural numbers. The natural

numbers themselves are a sequence[4] 1, 2, 3, ... which is arithmetic with $a = d = 1$. Applying Equation 9.2,

$$1 + 2 + 3 + \ldots + n \quad = \quad \frac{n(n+1)}{2}$$

So, for example, the sum of the first 100 natural numbers is $\frac{100(101)}{2} = 5050$.[5]

### 9.2.1 Exercises

1. Find the following sums.

   (a) $\sum_{g=4}^{9}(5g+3)$ (c) $\sum_{j=0}^{5}2^j$ (e) $\sum_{i=1}^{4}\frac{1}{4}(i^2+1)$

   (b) $\sum_{k=3}^{8}\frac{1}{k}$ (d) $\sum_{k=0}^{2}(3k-5)x^k$ (f) $\sum_{n=1}^{100}(-1)^n$

2. Rewrite the sum using summation notation.

   (a) $8 + 11 + 14 + 17 + 20$
   (b) $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$
   (c) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$
   (d) $1 + 2 + 4 + \cdots + 2^{29}$
   (e) $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5}$
   (f) $-\ln(3) + \ln(4) - \ln(5) + \cdots + \ln(20)$
   (g) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$
   (h) $\frac{1}{2}(x-5) + \frac{1}{4}(x-5)^2 + \frac{1}{6}(x-5)^3 + \frac{1}{8}(x-5)^4$

3. Find the sum of the first 10 terms of the following sequences.

   (a) $a_n = 3 + 5n$ (b) $b_n = \left(\frac{1}{2}\right)^n$ (c) $c_n = -2n + \left(\frac{5}{3}\right)^n$

4. Prove the properties listed in Theorem 9.1.

5. Discuss with your classmates the problems which arise in trying to find the sum of the following geometric series. When in doubt, write them out!

   (a) $\sum_{k=1}^{\infty}2^{k-1}$ (b) $\sum_{k=1}^{\infty}(1.0001)^{k-1}$ (c) $\sum_{k=1}^{\infty}(-1)^{k-1}$

### 9.2.2 Answers

1.

[4] This is the identity function on the natural numbers!

[5] There is an interesting anecdote which says that the famous mathematician Carl Friedrich Gauss was given this problem in primary school and devised a very clever solution.

(a) 213
(b) $\frac{341}{280}$
(c) 63
(d) $-5 - 2x + x^2$
(e) $\frac{17}{2}$
(f) 0

2. (a) $\sum_{k=1}^{5}(3k+5)$
(b) $\sum_{k=1}^{8}(-1)^{k-1}k$
(c) $\sum_{k=1}^{4}(-1)^{k-1}\frac{x}{2k-1}$
(d) $\sum_{k=1}^{29}2^{k-1}$
(e) $\sum_{k=1}^{5}\frac{k+1}{k}$
(f) $\sum_{k=3}^{20}(-1)^k\ln(k)$
(g) $\sum_{k=1}^{6}\frac{(-1)^{k-1}}{k^2}$
(h) $\sum_{k=1}^{4}\frac{1}{2k}(x-5)^k$

3. (a) 305
(b) $\frac{1023}{1024}$
(c) $\frac{17771050}{59049}$

4. (a) $\frac{7}{9}$
(b) $\frac{13}{99}$
(c) $\frac{3383}{333}$
(d) $-\frac{5809}{990}$

## 9.3 IRAs AND MORTGAGES

An important application of the geometric sum formula is the investment plan called an **Investment Retirement Account** (IRA). These differ from the kind of investments we studied in Section 7.5 in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6] Suppose you have an account with annual interest rate $r$ which is compounded $n$ times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. You will make ongoing deposits of $P$ dollars at the *end* of each compounding period. Let $A_k$ denote the amount in the account after $k$ compounding periods. Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on $A_1$ so that our initial investment has grown to $A_1(1+i) = P(1+i)$ in accordance with Equation 7.1. When we add our second payment at the end of the second period, we get

$$A_2 = A_1(1+i) + P = P(1+i) + P = P(1+i)\left(1 + \frac{1}{1+i}\right)$$

The reason for factoring out the $P(1+i)$ will become apparent in short order. During the third compounding period, we earn interest on $A_2$ which then grows to $A_2(1+i)$. We add our third

[6] The reader may wish to re-read the discussion on compound interest in Section 7.5 before proceeding.

payment at the end of the third compounding period to obtain

$$A_3 = A_2(1+i) + P = P(1+i)\left(1+\frac{1}{1+i}\right)(1+i) + P = P(1+i)^2\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, $A_3$ grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the $k$th compounding, we get

$$A_k = P(1+i)^{k-1}\left(1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first $k$ terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 9.2, we get

$$1+\frac{1}{1+i}+\frac{1}{(1+i)^2}+\ldots+\frac{1}{(1+i)^{k-1}} = 1\left(\frac{1-\frac{1}{(1+i)^k}}{1-\frac{1}{1+i}}\right) = \frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1}\left(\frac{(1+i)\left(1-(1+i)^{-k}\right)}{i}\right) = \frac{P\left((1+i)^k-1\right)}{i}$$

If we let $t$ be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the value after $t$ years.

Equation 9.3. **Future Value of an IRA:** Suppose an IRA offers an annual interest rate $r$ compounded $n$ times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit $P$ is made at the end of each compounding period, the amount $A$ in the account after $t$ years is given by

$$A = \frac{P\left((1+i)^{nt}-1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 9.3 and simplify. Some familiar equations arise which are cause for pause and meditation.

Example 9.3.1. An IRA offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the IRA in 30 years.

2. How many years will it take for the IRA to grow to \$100,000?

SOLUTION.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1 + 0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the IRA to grow to \$100,000, we set $A = 100000$ and solve for $t$. We isolate the exponential and take the natural logarithm of both sides of the equation.

$$\begin{aligned} 100000 &= \frac{50\left((1 + 0.005)^{12t} - 1\right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln\left((1.005)^{12t}\right) &= \ln(11) \\ 12t \ln(1.005) &= \ln(11) \\ t &= \frac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{aligned}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the IRA nearly doubles. This is a lesson worth remembering. □

A **mortgage** is like a backwards IRA. Here's an example.

EXAMPLE 9.3.2. A family takes out a 30 year \$100,000 mortgage to buy a house. The interest rate is 6% per year and they will repay the loan in 360 equal monthly payments. To compute the monthly payment imagine that the family has taken 360 loans

$$100000 = L_1 + L_2 + \cdots + L_{360}$$

and they will repay the $k$th loan at the end of the $k$th month with the monthly payment of $a$ dollars. The monthly interest rate is $6\%/12 = 0.005$ so the amount owed on the $k$th loan after $k$ months is $(1.005)^k L_k$. This is the amount of the $k$th payment so $a = (1.005)^k L_k$ so $L_k = a(1.005)^{-k}$ so

$$100000 = L_1 + L_2 + \cdots + L_{360} = a(1.005)^{-1} + a(1.005)^{-2} + \cdots + a(1.005)^{-360}.$$

Using Sigma notation and Theorem **??** with $r = (1.005)^{-1}$ this may be written

$$100000 = \sum_{k=1}^{360} a(1.005)^{-k} = a\frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1}$$

To evaluate the expression on the right multiply top and bottom by 1.005 to get

$$\frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1} = \frac{(1.005)^{-360} - 1}{1 - 1.005} = \frac{1 - (1.005)^{-360}}{0.005} = 166.79$$

so $100000 = 166.79a$ so the monthly payment is

$$a = 100000/166.79 = 599.56.$$

### 9.3.1 Exercises

1. If monthly payments of \$300 are made to an IRA with an APR of 2.5% compounded monthly what is the value of the IRA after 17 years?

2. Show that the formula for the future value of an IRA where the periodic payment is made at the *beginning* of the compounding period (rather than the end) is

$$A = P(1+i)\left[\frac{(1+i)^{nt} - 1}{i}\right]$$

### 9.3.2 Answers

1. \$76,163.67

## 9.4 Infinite sums and Repeating Decimals*

Definition 9.4. For a sequence $a_n$ the notation

$$\lim_{n\to\infty} a_n = b$$

means that that the numbers $a_n$ are arbitrarily close to the number $b$ when $n$ is sufficiently large. The expression on the left is called the the **limit** of $a_n$ as $n$ becomes infinite. For a series we also use the notation

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

THEOREM 9.2. If the ratio $r$ of successive terms in a geometric series is less than one in absolute value, then the sum of the infinite geometric series is

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}.$$

By Theorem **??**

$$\sum_{k=1}^{n} ar^k = \frac{ar - ar^{n+1}}{1-r} = \frac{ar}{1-r} + cr^n$$

where $c = ar/(1-r)$. In Section **??** we saw that for $|r| < 1$ the graph of the exponential function $y = r^x$ decays exponentially as $x$ becomes large positive. This implies that

$$\lim_{n\to\infty} r^n = 0$$

so

$$\sum_{k=1}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=1}^{n} ar^k = \frac{ar}{1-r} + c \lim_{n\to\infty} r^n = \frac{ar}{1-r}.$$

□

EXAMPLE 9.4.1. (Zeno's Paradox) To travel one mile I must first travel the first half mile, then half of the remaining distance, then half of the remaining distance, and so on. How can I ever go whole distance? The answer is that the sum of all the distances is one:

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4}, \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \qquad \ldots.$$

The numerator is only one less than the denominator:

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n} = 1 - 2^{-n}.$$

The finite sums are getting closer to one and the infinite sum is

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

Example 9.4.2. We use Theorem 9.2 to prove that the infinite repeating decimal $0.1\,36\,36\,36\,36\,36\,36\ldots$ is equal to $3/22$. The first step is to make the decimal look like a geometric series.

$$\begin{aligned}
0.1\,36\,36\,36\,36\,36\,36\ldots &= \frac{1}{10} + 36\big(0.0\,01\,01\,01\,01\,01\ldots\big) \\
&= \frac{1}{10} + \frac{36}{10}\big(0.01\,01\,01\,01\,01\ldots\big) \\
&= \frac{1}{10} + \frac{36}{10}\big(10^{-2} + 10^{-4} + 10^{-6} + 10^{-8} + \cdots\big) \\
&= \frac{1}{10} + \frac{36}{10}\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^k
\end{aligned}$$

Now by Theorem **??**

$$\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^k = \frac{0.01}{1-0.01} = \frac{0.01}{0.99} = \frac{1}{99}$$

so

$$0.1\,36\,36\,36\,36\ldots = \frac{1}{10} + \frac{36}{10}\cdot\frac{1}{99}.$$

Now we do the arithmetic:

$$\frac{1}{10} + \frac{36}{10}\cdot\frac{1}{99} = \frac{99+36}{990} = \frac{135}{990} = \frac{15}{110} = \frac{3}{22}.$$

The concept of an infinite sum makes the definition of decimal expansion more precise. If $x$ is a real number between 0 and 1 it has a decimal expansion

$$x = \sum_{k=1}^{\infty} d_k\, 10^{-k}$$

where each $d_k$ is an integer between 0 and 9. For most real numbers the digits $d_k$ won't follow any pattern, but

Theorem 9.3. A real number $x$ is rational if and only if has a repeating decimal expansion like the one in Example 9.4.2.

A rational number is a ratio $p/q$ of two integers $p$ and $q$. To see why the decimal expansion eventually repeats periodically imagine computing the decimal expansion by long division. At each step in the long division algorithm we compute the next digit in the quotient, multiply that digit by $q$, subtract the product to get the next remainder, and bring down the next digit from the dividend. The remainder is smaller than $q$, otherwise we would have used a larger digit in the quotient we are computing. Once we are computing digits of the right of the decimal point the digit we bring down from the dividend is always zero and since the remainder is always less than $q$ we will eventually find ourselves redoing what we have already done.

As an example we express 1202/99 as a repeating decimal. We do long division as in elementary school.

$$\begin{array}{rl} & 12.141 \\ \hline 99 \;\big|\; & 1202.000 \\ & \underline{99} \\ & \overline{212} \\ & \underline{188} \\ & \overline{140} \\ & \underline{99} \\ & \overline{410} \\ & \underline{396} \\ & 140 \end{array} \begin{array}{l} \\ \\ \\ \\ 0 \quad \leftarrow \\ 9 \\ \\ \\ \leftarrow \end{array}$$

After we are to the right of the decimal point we always bring down a zero and the two rows marked with an arrow have the same remainder. Hence the process must cycle (repeat periodically) at that point (every other time we do the same division and subtraction) so

$$\frac{1202}{99} = 12.\overline{14} := 12.\,14\,14\,14\ldots$$

The proof that a real number with a repeating decimal expansion is rational is just like this last computation. We first write the number as the sum of a finite decimal and a negative power of ten times a geometric series. We then use Theorem 9.2 and do the arithmetic. □

### 9.4.1 EXERCISES

1. Express the following repeating decimals as a fraction of integers.

   (a) $0.\overline{7}$ (b) $0.\overline{13}$ (c) $10.\overline{159}$ (d) $-5.8\overline{67}$

# Chapter 10

# Complex Numbers and the Fundamental Theorem of Algebra

## 10.1 Complex Numbers

Consider the polynomial $p(x) = x^2 + 1$. The zeros of $p$ are the solutions to $x^2 + 1 = 0$, or $x^2 = -1$. This equation has no real solutions, but you may recall from Intermediate Algebra that we can formally extract the square roots of both sides to get $x = \pm\sqrt{-1}$. The quantity $\sqrt{-1}$ is usually relabeled $i$, the so-called **imaginary unit**.[1] The number $i$, while not a real number, plays along well with real numbers, and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2 - 7i) + (3 + 4i) = 5 - 3i$, and so forth. The key properties which distinguish $i$ from the real numbers are listed below.

Definition 10.1. The imaginary unit $i$ satisfies the two following properties

1. $i^2 = -1$
2. If $c$ is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Property 1 in Definition 10.1 establishes that $i$ does act as a square root[2] of $-1$, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on $c$. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

[1] Some technical mathematics textbooks label it '$j$'.

[2] Note the use of the indefinite article 'a'. Whatever beast is chosen to be $i$, $-i$ is the other square root of $-1$.

which is unacceptable.[3] We are now in the position to define the **complex numbers**.

DEFINITION 10.2. A **complex number** is a number of the form $a+bi$, where $a$ and $b$ are real numbers and $i$ is the imaginary unit.

Complex numbers include things you'd normally expect, like $3+2i$ and $\frac{2}{5}-i\sqrt{3}$. However, don't forget that $a$ or $b$ could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers **include** the real numbers, so 0 and $\pi-\sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only thing you need to remember are the two properties in Definition 10.1. The next example should help recall how these animals behave.

EXAMPLE 10.1.1. Perform the indicated operations and simplify. Write your final answer in the form[4] $a+bi$.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

SOLUTION.

1. As mentioned earlier, we treat expressions involving $i$ as we would any other radical. We combine like terms to get $(1-2i)-(3+4i)=1-2i-3-4i=-2-6i$.
2. Using the distributive property, we get $(1-2i)(3+4i)=(1)(3)+(1)(4i)-(2i)(3)-(2i)(4i)=3+4i-6i-8i^2$. Recalling $i^2=-1$, we get $3+4i-6i-8i^2=3-2i-(-8)=11-2i$.
3. How in the world are we supposed to simplify $\frac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing a radical, and multiply both numerator and denominator by $3+4i$ (the conjugate of $3-4i$).[5] Doing so produces

$$\frac{1-2i}{3-4i}\cdot\frac{3+4i}{3+4i}=\frac{(1-2i)(3+4i)}{(3-4i)(3+4i)}=\frac{11-2i}{25}=\frac{11}{25}-\frac{2}{25}\,i$$

4. We use property 2 of Definition 10.1 first, then apply the rules of radicals applicable to real radicals to get $\sqrt{-3}\sqrt{-12}=\left(i\sqrt{3}\right)\left(i\sqrt{12}\right)=i^2\sqrt{3\cdot 12}=-\sqrt{36}=-6$.

[3] We want to enlarge the number system so we can solve things like $x^2=-1$, but not at the cost of the established rules already set in place. For that reason, the general properties of radicals simply do not apply for even roots of negative quantities.

[4] We'll accept an answer of say $3-2i$, although, technically, we should write this as $3+(-2)i$. Even we pedants have our limits.

[5] We will talk more about this in a moment.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned}(x-[1+2i])(x-[1-2i]) &= x^2 - x[1-2i] - x[1+2i] + [1-2i][1+2i] \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 \\ &= x^2 - 2x + 5\end{aligned}$$

□

A couple of remarks about the last example are in order. First, the **conjugate** of a complex number $a + bi$ is the number $a - bi$. The notation commonly used for conjugation is a 'bar': $\overline{a+bi} = a-bi$. For example, $\overline{3+2i} = 3-2i$, $\overline{3-2i} = 3+2i$, $\overline{6} = 6$, $\overline{4i} = -4i$, and $\overline{3+\sqrt{5}} = 3+\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem 10.1. Suppose $z$ and $w$ are complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z} + \overline{w} = \overline{z+w}$
- $\overline{z}\,\overline{w} = \overline{zw}$
- $(\overline{z})^n = \overline{z^n}$, for any natural number $n = 1, 2, 3, \ldots$
- $z$ is a real number if and only if $\overline{z} = z$.

Essentially, Theorem 10.1 says that complex conjugation works well with addition, multiplication, and powers. The proof of these properties can best be achieved by writing out $z = a + bi$ and $w = c + di$ for real numbers $a$, $b$, $c$, and $d$. Next, we compute the left and right hand side of each equation and check to see that they are the same. The proof of the first property is a very quick exercise.[6] To prove the second property, we compare $\overline{z} + \overline{w}$ and $\overline{z+w}$. We have $\overline{z} + \overline{w} = \overline{a+bi} + \overline{c+di} = a - bi + c - di$. To find $\overline{z+w}$, we first compute

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

so

$$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c) - (b+d)i = a - bi + c - di$$

[6] Trust us on this.

As such, we have established $\overline{z}+\overline{w} = \overline{z+w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction.[7] The last property is a characterization of real numbers. If $z$ is real, then $z = a + 0i$, so $\overline{z} = a - 0i = a = z$. On the other hand, if $z = \overline{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

## 10.2 THE FUNDAMENTAL THEOREM OF ALGEBRA

We now return to the business of zeros. Suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note the quadratic doesn't factor nicely, so we resort to the Quadratic Formula, Equation 3.5 and obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros, $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, we note that in Example 10.1.1, part 6, we found $(x - [1 + 2i])(x - [1 - 2i]) = x^2 - 2x + 5$. This demonstrates that the factor theorem holds even for non-real zeros, i.e, $x = 1 + 2i$ is a zero of $f$, and, sure enough, $(x - [1 + 2i])$ is a factor of $f(x)$. It turns out that polynomial division works the same way for all complex numbers, real and non-real alike, and so the Factor and Remainder Theorems hold as well. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

THEOREM 10.2. **The Fundamental Theorem of Algebra:** Suppose $f$ is a polynomial function with complex number coefficients of degree $n \geq 1$, then $f$ has least one complex zero.

The Fundamental Theorem of Algebra is an example of an 'existence' theorem in mathematics. Like the Intermediate Value Theorem, Theorem 4.1, the Fundamental Theorem of Algebra guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. In fact, there are polynomials whose real zeros, though they exist, cannot be expressed using the 'usual' combinations of arithmetic symbols, and must be approximated. The authors are fully aware that the full impact and profound nature of the Fundamental Theorem of Algebra is lost on most students this level, and that's fine. It took mathematicians literally hundreds of years to prove the theorem in its full generality, and some of that history is recorded here. Note that the Fundamental Theorem of Algebra applies to polynomial functions with not only real coefficients, but, those with complex number coefficients as well.

[7] See Section **??**.

Suppose $f$ is a polynomial of degree $n$ with $n \geq 1$. The Fundamental Theorem of Algebra guarantees us at least one complex zero, $z_1$, and, as such, the Factor Theorem guarantees that $f(x)$ factors as $f(x) = (x - z_1)\, q_1(x)$ for a polynomial function $q_1$, of degree exactly $n - 1$. If $n - 1 \geq 1$, then the Fundamental Theorem of Algebra guarantees a complex zero of $q_1$ as well, say $z_2$, and so the Factor Theorem gives us $q_1(x) = (x - z_2)\, q_2(x)$, and hence $f(x) = (x - z_1)\,(x - z_2)\, q_2(x)$. We can continue this process exactly $n$ times, at which point our quotient polynomial $q_n$ has degree 0 so it's a constant. This argument gives us the following factorization theorem.

Theorem 10.3. **Complex Factorization Theorem:** Suppose $f$ is a polynomial function with complex number coefficients. If the degree of $f$ is $n$ and $n \geq 1$, then $f$ has exactly $n$ complex zeros, counting multiplicity. If $z_1$, $z_2$, ..., $z_k$ are the distinct zeros of $f$, with multiplicities $m_1$, $m_2$, ..., $m_k$, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Note that the value $a$ in Theorem 10.3 is the leading coefficient of $f(x)$ (Can you see why?) and as such, we see that a polynomial is completely determined by its zeros, their multiplicities, and its leading coefficient. We put this theorem to good use in the next example.

Example 10.2.1. Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

1. Find all complex zeros of $f$ and state their multiplicities.
2. Factor $f(x)$ using Theorem 10.3

Solution.

1. Since $f$ is a fifth degree polynomial, we know we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. We get

$$\begin{array}{r|cccccc}
\frac{1}{2} & 12 & -20 & 19 & -6 & -2 & 1 \\
 & \downarrow & 6 & -7 & 6 & 0 & -1 \\
\hline
\frac{1}{2} & 12 & -14 & 12 & 0 & -2 & \boxed{0} \\
 & \downarrow & 6 & -4 & 4 & 2 & \\
\hline
-\frac{1}{3} & 12 & -8 & 8 & 4 & \boxed{0} & \\
 & \downarrow & -4 & 4 & -4 & & \\
\hline
 & 12 & -12 & 12 & \boxed{0} & &
\end{array}$$

Our quotient is $12x^2 - 12x + 12$, whose zeros we find to be $\frac{1 \pm i\sqrt{3}}{2}$. From Theorem 10.3, we know $f$ has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

2. Applying Theorem 10.3, we are guaranteed that $f$ factors as

$$f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(x - \left[\frac{1+i\sqrt{3}}{2}\right]\right)\left(x - \left[\frac{1-i\sqrt{3}}{2}\right]\right)$$

□

A true test of Theorem 10.3 (and a student's mettle!) would be to take the factored form of $f(x)$ in the previous example and multiply it out[8] to see that it really does reduce to the formula $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$. When factoring a polynomial using Theorem 10.3, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the **real numbers** then we would have stopped short of finding the nonreal zeros of $f$ and factored $f$ using our work from the synthetic division to write $f(x) = \left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(12x^2 - 12x + 12\right)$, or $f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(x^2 - x + 1\right)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using **real** numbers. The last two results of the section show us that, at least in theory, if we have a polynomial function with real coefficients, we can always factor it down enough so that any nonreal zeros come from irreducible quadratics.

THEOREM 10.4. **Conjugate Pairs Theorem:** If $f$ is a polynomial function with real number coefficients and $z$ is a zero of $f$, then so is $\overline{z}$.

To prove the theorem, suppose $f$ is a polynomial with real number coefficients. Specifically, let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$. If $z$ is a zero of $f$, then $f(z) = 0$, which means $a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 = 0$. Next, we consider $f\left(\overline{z}\right)$ and apply Theorem 10.1 below.

[8] You really should do this once in your life to convince yourself that all of the theory actually does work!

$$\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \ldots + a_2(\overline{z})^2 + a_1\overline{z} + a_0 \\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\overline{z} + a_0 && \text{since } (\overline{z})^n = \overline{z^n} \\
&= \overline{a_n}\overline{z^n} + \overline{a_{n-1}}\overline{z^{n-1}} + \ldots + \overline{a_2}\overline{z^2} + \overline{a_1}\,\overline{z} + \overline{a_0} && \text{since the coefficients are real} \\
&= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} && \text{since } \overline{z}\,\overline{w} = \overline{zw} \\
&= \overline{a_n z^n + a_{n-1}z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0} && \text{since } \overline{z} + \overline{w} = \overline{z+w} \\
&= \overline{f(z)} \\
&= \overline{0} \\
&= 0
\end{aligned}$$

This shows that $\overline{z}$ is a zero of $f$. So, if $f$ is a polynomial function with real number coefficients, Theorem 10.4 tells us if $a + bi$ is a nonreal zero of $f$, then so is $a - bi$. In other words, nonreal zeros of $f$ come in conjugate pairs. The Factor Theorem kicks in to give us both $(x - [a + bi])$ and $(x - [a - bi])$ as factors of $f(x)$ which means $(x - [a + bi])(x - [a - bi]) = x^2 + 2ax + \left(a^2 + b^2\right)$ is an irreducible quadratic factor of $f$. As a result, we have our last result of the section.

Theorem 10.5. **Real Factorization Theorem:** Suppose $f$ is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of $f$ and irreducible quadratic factors which give the nonreal zeros of $f$.

We now present an example which pulls together all of the major ideas of this section.

Example 10.2.2. Let $f(x) = x^4 + 64$.

1. Use synthetic division to show $x = 2 + 2i$ is a zero of $f$.
2. Find the remaining complex zeros of $f$.
3. Completely factor $f(x)$ over the complex numbers.
4. Completely factor $f(x)$ over the real numbers.

Solution.

1. Remembering to insert the 0's in the synthetic division tableau we have

$$\begin{array}{r|cccc} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline & 1 & 2+2i & 8i & -16+16i & \boxed{0} \end{array}$$

2. Since $f$ is a fourth degree polynomial, we need to make two successful divisions to get a quadratic quotient. Since $2+2i$ is a zero, we know from Theorem 10.4 that $2-2i$ is also a zero. We continue our synthetic division tableau.

$$\begin{array}{r|cccc} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline 2-2i & 1 & 2+2i & 8i & -16+16i & \boxed{0} \\ & \downarrow & 2-2i & 8-8i & 16-16i & \\ \hline & 1 & 4 & 8 & \boxed{0} & \end{array}$$

Our quotient polynomial is $x^2+4x+8$. Using the quadratic formula, we obtain the remaining zeros $-2+2i$ and $-2-2i$.

3. Using Theorem 10.3, we get $f(x) = (x-[2-2i])(x-[2+2i])(x-[-2+2i])(x-[-2-2i])$.

4. We multiply the linear factors of $f(x)$ which correspond to complex conjugate pairs. We find $(x-[2-2i])(x-[2+2i]) = x^2-4x+8$, and $(x-[-2+2i])(x-[-2-2i]) = x^2+4x+8$. Our final answer $f(x) = \left(x^2-4x+8\right)\left(x^2+4x+8\right)$. □

Our last example turns the tables and asks us to manufacture a polynomial with certain properties of its graph and zeros.

EXAMPLE 10.2.3. Find a polynomial $p$ of lowest degree that has integer coefficients and satisfies all of the following criteria:

- the graph of $y = p(x)$ touches the $x$-axis at $\left(\frac{1}{3}, 0\right)$
- $x = 3i$ is a zero of $p$.
- as $x \to -\infty$, $p(x) \to -\infty$
- as $x \to \infty$, $p(x) \to -\infty$

SOLUTION. To solve this problem, we will need a good understanding of the relationship between the $x$-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behavior of

polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of $p$ touches the $x$-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2. The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x-3i)$ and $(x+3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x - 3i)(x + 3i)$ for some real number $a$. Expanding this, we get $p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$. In order to obtain integer coefficients, we know $a$ must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is $ax^4$, we need $a < 0$ to get $p(x) \to -\infty$ as $x \to \pm\infty$. Hence, if we choose $x = -9$, we get $p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$. We can verify our handiwork using the techniques developed in this chapter. □

This example concludes our study of polynomial functions.[9] The last few sections have contained what is considered by many to be 'heavy' mathematics. Like a heavy meal, heavy mathematics takes time to digest. Don't be overly concerned if it doesn't seem to sink in all at once, and pace yourself on the exercises or you're liable to get mental cramps. But before we get to the exercises, we'd like to offer a bit of an epilogue.

Our main goal in presenting the material on the complex zeros of a polynomial was to give the chapter a sense of completeness. Given that it can be shown that some polynomials have real zeros which cannot be expressed using the usual algebraic operations, and still others have no real zeros at all, it was nice to discover that every polynomial of degree $n \geq 1$ has $n$ complex zeros. So like we said, it gives us a sense of closure. But the observant reader will note that we did not give any examples of applications which involve complex numbers. Students often wonder when complex numbers will be used in 'real-world' applications. After all, didn't we call $i$ the imaginary unit? How can imaginary things be used in reality? It turns out that complex numbers are very useful in many applied fields such as fluid dynamics, electromagnetism and quantum mechanics, but most of the applications require Mathematics well beyond College Algebra to fully understand them. That does not mean you'll never be be able to understand them; in fact, it is the authors' sincere hope that all of you will reach a point in your studies when the glory, awe and splendor of complex numbers are revealed to you. For now, however, the really good stuff is beyond the scope of this text. We invite you and your classmates to find a few examples of complex number applications and see what you can make of them. A simple Internet search with the phrase 'complex numbers in real life' should get you started. Basic electronics classes are another place to look, but remember, they might use the letter $j$ where we have used $i$.

For the remainder of the text, we will restrict our attention to real numbers. We do this primarily because the first Calculus sequence you will take, ostensibly the one that this text is preparing you for, studies only functions of real variables. Also, lots of really cool scientific things don't require any deep understanding of complex numbers to study them, but they do need more Mathematics like exponential, logarithmic and trigonometric functions. We believe it makes more

[9] With the exception of the exercises on the next page, of course.

sense pedagogically for you to learn about those functions now and then, after you've completed the Calculus sequence, take a course in Complex Function Theory in your junior or senior year. It is in that course that the true power of the complex numbers is released. But for now, in order to fully prepare you for life immediately after College Algebra, we will say that functions like $f(x) = \dfrac{1}{x^2+1}$ have a domain of all real numbers, even though we know $x^2 + 1 = 0$ has two complex solutions, namely $x = \pm i$. Because $x^2 + 1 > 0$ for all *real* numbers $x$, the fraction $\frac{1}{x^2+1}$ is never undefined in our real variable setting.

### 10.2.1 Exercises

1. We know $i^2 = -1$ which means $i^3 = i^2 \cdot i = (-1) \cdot i = -i$ and $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$. Use this information to simplify the following.

   (a) $i^5$ (b) $i^{304}$ (c) $(2i)^3$ (d) $(-i)^{23}$

2. Let $z = 3 + 4i$ and $w = 2 - i$. Compute the following and express your answer in $a + bi$ form.

   (a) $z + w$ (c) $z \cdot w$ (e) $\dfrac{w}{z}$

   (b) $w - z$ (d) $\dfrac{z}{w}$ (f) $w^3$

3. Simplify the following.

   (a) $\sqrt{-49}$ (c) $\sqrt{(-9)(-16)}$

   (b) $\sqrt{-9}\sqrt{-16}$ (d) $\sqrt{49}\sqrt{-4}$

4. Find the complex solutions of the following quadratic equations.

   (a) $x^2 - 4x + 13 = 0$ (b) $3x^2 + 2x + 10 = 0$

5. For each polynomial given below find all of its zeros, completely factor it over the real numbers and completely factor it over the complex numbers.

   (a) $x^2 - 2x + 5$
   (b) $x^3 - 2x^2 + 9x - 18$
   (c) $x^3 + 6x^2 + 6x + 5$
   (d) $3x^3 - 13x^2 + 43x - 13$
   (e) $x^4 + 9x^2 + 20$
   (f) $4x^4 - 4x^3 + 13x^2 - 12x + 3$
   (g) $x^3 + 3x^2 + 4x + 12$
   (h) $2x^4 - 7x^3 + 14x^2 - 15x + 6$
   (i) $4x^3 - 6x^2 - 8x + 15$
   (j) $x^4 + x^3 + 7x^2 + 9x - 18$
   (k) $6x^4 + 17x^3 - 55x^2 + 16x + 12$
   (l) $x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
   (m) $x^3 + 7x^2 + 9x - 2$
   (n) $-3x^4 - 8x^3 - 12x^2 - 12x - 5$
   (o) $8x^4 + 50x^3 + 43x^2 + 2x - 4$
   (p) $9x^3 + 2x + 1$
   (q) $x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
   (r) $2x^4 + 5x^3 + 13x^2 + 7x + 5$ (Hint: $x = -1 + 2i$ is a zero.)

6. Let $z$ and $w$ be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

7. With the help of your classmates, build a polynomial $p$ with integer coefficients such that $x = -2 - i$ is a zero of $p$, $p$ has a local maximum at the point $(4, 0)$ and $p(x) \to -\infty$ as $x \to \infty$

### 10.2.2 ANSWERS

1. (a) $i^5 = i^4 \cdot i = 1 \cdot i = i$
   (b) $i^{304} = (i^4)^{76} = 1^{76} = 1$
   (c) $(2i)^3 = 8i^3 = -8i$
   (d) $(-i)^{23} = -i^{23} = -i^{20} \cdot i^3 = (-1)(-i) = i$

2. (a) $z + w = 5 + 3i$
   (b) $w - z = -1 - 5i$
   (c) $z \cdot w = 10 + 5i$
   (d) $\dfrac{z}{w} = \dfrac{2}{5} + \dfrac{11}{5}i$
   (e) $\dfrac{w}{z} = \dfrac{2}{25} - \dfrac{11}{25}i$
   (f) $w^3 = 2 - 11i$

3. (a) $\sqrt{-49} = 7i$
   (b) $\sqrt{-9}\sqrt{-16} = (3i) \cdot (4i) = 12i^2 = -12$
   (c) $\sqrt{(-9)(-16)} = \sqrt{144} = 12$
   (d) $\sqrt{49}\sqrt{-4} = 7 \cdot 2i = 14i$

4. (a) $x = 2 \pm 3i$
   (b) $x = -\dfrac{1}{3} \pm \dfrac{\sqrt{29}}{3}i$

5. (a) $x^2 - 2x + 5 = (x - (1 + 2i))(x - (1 - 2i))$
   Zeros: $x = 1 \pm 2i$
   (b) $x^3 - 2x^2 + 9x - 18 = (x - 2)\left(x^2 + 9\right) = (x - 2)(x - 3i)(x + 3i)$
   Zeros: $x = 2, \pm 3i$
   (c) $x^3+6x^2+6x+5 = (x+5)(x^2+x+1) = (x+5)\left(x - \left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)\right)$
   Zeros: $x = -5,\ x = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$
   (d) $3x^3 - 13x^2 + 43x - 13 = (3x - 1)(x^2 - 4x + 13) = (3x - 1)(x - (2 + 3i))(x - (2 - 3i))$
   Zeros: $x = \dfrac{1}{3},\ x = 2 \pm 3i$
   (e) $x^4 + 9x^2 + 20 = \left(x^2 + 4\right)\left(x^2 + 5\right) = (x - 2i)(x + 2i)\left(x - i\sqrt{5}\right)\left(x + i\sqrt{5}\right)$
   Zeros: $x = \pm 2i, \pm i\sqrt{5}$
   (f) $4x^4 - 4x^3 + 13x^2 - 12x + 3 = (2x - 1)^2(x^2 + 3) = (2x - 1)^2(x + \sqrt{3}i)(x - \sqrt{3}i)$
   Zeros: $x = \dfrac{1}{2},\ x = \pm\sqrt{3}i$
   (g) $x^3 + 3x^2 + 4x + 12 = (x + 3)\left(x^2 + 4\right) = (x + 3)(x + 2i)(x - 2i)$
   Zeros: $x = -3,\ \pm 2i$
   (h) $2x^4 - 7x^3 + 14x^2 - 15x + 6 = (x - 1)^2\left(2x^2 - 3x + 6\right)$
   $= 2(x - 1)^2\left(x - \left(\dfrac{3}{4} + \dfrac{\sqrt{39}}{4}i\right)\right)\left(x - \left(\dfrac{3}{4} - \dfrac{\sqrt{39}}{4}i\right)\right)$
   Zeros: $x = 1,\ x = \dfrac{3}{4} \pm \dfrac{\sqrt{39}}{4}i$

(i) $4x^3 - 6x^2 - 8x + 15 = \left(x + \frac{3}{2}\right)\left(4x^2 - 12x + 10\right) =$
$4\left(x + \frac{3}{2}\right)\left(x - \left(\frac{3}{2} + \frac{1}{2}i\right)\right)\left(x - \left(\frac{3}{2} - \frac{1}{2}i\right)\right)$
Zeros: $x = -\frac{3}{2},\ x = \frac{3}{2} \pm \frac{1}{2}i$

(j) $x^4 + x^3 + 7x^2 + 9x - 18 = (x+2)(x-1)\left(x^2+9\right) = (x+2)(x-1)(x+3i)(x-3i)$
Zeros: $x = -2,\ 1,\ \pm 3i$

(k) $6x^4+17x^3-55x^2+16x+12 = 6\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right)\left(x - \left(-2 + 2\sqrt{2}\right)\right)\left(x - \left(-2 - 2\sqrt{2}\right)\right)$
Zeros: $x = -\frac{1}{3},\ x = \frac{3}{2},\ x = -2 \pm 2\sqrt{2}$

(l) $x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x-1)\left(x^2+3\right)\left(x^2+4\right)$
$= (x-1)(x - i\sqrt{3})(x + i\sqrt{3})(x-2i)(x+2i)$
Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

(m) $x^3 + 7x^2 + 9x - 2 = (x+2)\left(x - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(x - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
Zeros: $x = -2,\ x = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

(n) $-3x^4 - 8x^3 - 12x^2 - 12x - 5 = (x+1)^2\left(-3x^2 - 2x - 5\right)$
$= -3(x+1)^2\left(x - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$
Zeros: $x = -1,\ x = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

(o) $8x^4 + 50x^3 + 43x^2 + 2x - 4 = 8\left(x + \frac{1}{2}\right)\left(x - \frac{1}{4}\right)(x - (-3 + \sqrt{5}))(x - (-3 - \sqrt{5}))$
Zeros: $x = -\frac{1}{2},\ \frac{1}{4},\ x = -3 \pm \sqrt{5}$

(p) $9x^3 + 2x + 1 = \left(x + \frac{1}{3}\right)\left(9x^2 - 3x + 3\right)$
$= 9\left(x + \frac{1}{3}\right)\left(x - \left(\frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\right)\left(x - \left(\frac{1}{6} - \frac{\sqrt{11}}{6}i\right)\right)$
Zeros: $x = -\frac{1}{3},\ x = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$

(q) $x^4-2x^3+27x^2-2x+26 = (x^2-2x+26)(x^2+1) = (x-(1+5i))(x-(1-5i))(x+i)(x-i)$
Zeros: $x = 1 \pm 5i,\ x = \pm i$

(r) $2x^4 + 5x^3 + 13x^2 + 7x + 5 = \left(x^2 + 2x + 5\right)\left(2x^2 + x + 1\right) =$
$2(x - (-1 + 2i))(x - (-1 - 2i))\left(x - \left(-\frac{1}{4} + i\frac{\sqrt{7}}{4}\right)\right)\left(x - \left(-\frac{1}{4} - i\frac{\sqrt{7}}{4}\right)\right)$
Zeros: $x = -1 \pm 2i, -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

# Appendix A

# The Laws of Algebra Proved

## A.1 The Laws of Algebra

**The Laws of Algebra.**

All the rules of calculation that you learned in elementary school follow from the above fundamental laws. In particular, the Commutative and Associative Laws say that you can add a bunch of numbers in any order and similarly you can multiply a bunch of numbers in any order. For example,

$$(A+B)+(C+D)=(A+C)+(B+D), \qquad (A\cdot B)\cdot(C\cdot D)=(A\cdot C)\cdot(B\cdot D).$$

## A.2 The Analogy between Addition and Multiplication

Because both addition and multiplication satisfy the commutative, associative, identity, and inverse laws, there are other analogies:

Theorem A.1. Inverses are unique: $b+a=0 \implies b=-a$; similarly $b\cdot a=1 \implies b=a^{-1}$.

*Proof.* Assume that $b+a=0$. Then

| step | by | with |
|---|---|---|
| $b=b+0$ | $A=A+0$ | $A=b$ |
| $=b+\big(a+(-a)\big)$ | $A+(-A)=0$ | $A=a$ |
| $=\big(b+a\big)+(-a)$ | $A+(B+C)=(A+B)+C$ | $A=b,\ B=a,\ C=-a$ |
| $=0+(-a)$ | $b+a=0$ | by hypothesis |
| $=-a$ | $0+A=A$ | $A=-a$ |

Similarly assume that $b\cdot a^{-1}=1$. Then

| step | by | with |
|---|---|---|
| $b = b \cdot 1$ | $A = A \cdot 1$ | $A = b$ |
| $= b \cdot \left(a \cdot a^{-1}\right)$ | $A \cdot A^{-1} = 1$ | $A = a$ |
| $= \left(b \cdot a\right) \cdot a^{-1}$ | $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ | $A = b$, $B = a$, $C = a^{-1}$ |
| $= 1 \cdot a^{-1}$ | $b \cdot a = 1$ | by hypothesis |
| $= a^{-1}$ | $1 \cdot A = A$ | $A = a^{-1}$ |

□

Here are the proofs of the properties (i-vi) above. Notice how each proof of an addition rule is immediately followed by the proof of the corresponding multiplication rule.

THEOREM A.2. $-(-a) = a$ and similarly $(a^{-1})^{-1} = a$.

*Proof.*

| step | by | with |
|---|---|---|
| $a + (-a) = 0$ | $A + (-A) = 0$ | $A = a$ |
| $(-a) + a = 0$ | $A + B = B + A$ | $A = a$, $B = -a$ |
| $a = -(-a)$ | $A + B = 0 \implies B = -A$ | $A = -a$, $B = a$ |

| step | by | with |
|---|---|---|
| $a \cdot a^{-1} = A$ | $A \cdot A^{-1}$ | $A = a$ |
| $a^{-1} \cdot a = 1$ | $A \cdot B = B \cdot A$ | $A = a$, $B = a^{-1}$ |
| $a = -(-a)$ | $A \cdot B = 1 \implies B = A^{-1}$ | $A = a^{-1}$, $B = a$ |

□

THEOREM A.3. $-(a + b) = -a - b$ and similarly $(ab)^{-1} = a^{-1}b^{-1}$.

*Proof.*

| step | by | with |
|---|---|---|
| $(a+b)+(-a-b)$ | | |
| $\quad=(a+b)+\big((-a)+(-b)\big)$ | $A-B):=A+(-B)$ | $A=-a,\ B=b$ |
| $\quad=\big(a+(-a)\big)+\big(b+(-b)\big)$ | $(A+B)+(C+D)=$ | $A=a,\ B=b,$ |
| | $\quad=(A+C)+(B+D)$ | $\quad C=-a,\ D=-b$ |
| $\quad=\big(a+(-a)\big)+0$ | $A+(-A)=0$ | $A=b$ |
| $\quad=0+0$ | $A+(-A)=0$ | $A=a$ |
| $\quad=0$ | $A+0=A$ | $A=0$ |
| $-a-b=-(a+b)$ | $A+B=0\implies B=-A$ | $A=(a+b),\ B=-a-b$ |

| step | by | with |
|---|---|---|
| $(a\cdot b)\cdot\big((a^{-1})\cdot(b^{-1})\big)=$ | | |
| $\quad=\big(a\cdot a^{-1}\big)\cdot\big(b\cdot b^{-1}\big)$ | $(A\cdot B)\cdot(C\cdot D)=$ | $A=a,\ B=b,$ |
| | $\quad=(A\cdot C)\cdot(B\cdot D)$ | $C=a^{-1},\ D=b^{-1}$ |
| $\quad=\big(a\cdot a^{-1}\big)\cdot 1$ | $A\cdot A^{-1}$ | $A=b$ |
| $\quad=1\cdot 1$ | $A\cdot A^{-1}$ | $A=a$ |
| $\quad=1$ | $A\cdot 1=A$ | $A=1$ |
| $(a\cdot b)^{-1}=a^{-1}\cdot b^{-1}$ | $A\cdot B=1\implies B=A^{-1}$ | $A=a^{-1}\cdot b^{-1},\ B=a\cdot b$ |

□

THEOREM A.4. $-(a-b) = b-a$ and similarly $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

*Proof.*

| step | by | with |
|---|---|---|
| $-(a-b) = -\big(a+(-b)\big)$ | $A-B := A+(-B)$ | $A=a,\ B=b$ |
| $= -a-(-b)$ | $-(A+B) = -A-B$ | $A=a,\ B=-b$ |
| $= -a+(-(-b))$ | $A-B := A+(-B)$ | $A=a,\ B=-b$ |
| $= -a+b$ | $-(-A) = A$ | $A=b$ |
| $= b+(-a)$ | $A+B = B+A$ | $A=-a,\ B=b$ |
| $= b-a$ | $A+(-B) =: A-B$ | $A=b,\ B=a$ |

| step | by | with |
|---|---|---|
| $(a/b)^{-1} = \big(a\cdot b^{-1}\big)^{-1}$ | $A/B := A\cdot B^{-1}$ | $A=a,\ B=b$ |
| $= a^{-1}\cdot(b^{-1})^{-1}$ | $(A\cdot B)^{-1} = A^{-1}\cdot B^{-1}$ | $A=a,\ B=b^{-1}$ |
| $= a^{-1}\cdot b$ | $(A^{-1})^{-1} = A$ | $A=b$ |
| $= b\cdot a^{-1}$ | $A\cdot B = B\cdot A$ | $A=a^{-1},\ B=b$ |
| $= b/a$ | $A\cdot B^{-1} =: A/B$ | $A=b,\ B=a$ |

□

THEOREM A.5. $(a-b)+(c-d) = (a+c)-(b+d)$ and similarly $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$.

*Proof.*

| step | by | with |
|---|---|---|
| $(a-b)+(c-d)$ | | |
| $= \big(a+(-b)\big)+(c-d)$ | $A-B := A+(-B)$ | $A=a,\ B=b$ |
| $= \big(a+(-b)\big)+\big(c+(-d)\big)$ | $A-B := A+(-B)$ | $A=c,\ B=d$ |
| $= \big(a+c\big)+\big((-b)+(-d)\big)$ | $(A+B)+(C+D) =$ $= (A+C)+(B+D)$ | $A=a,\ B=-b,$ $C=c,\ D=-d$ |
| $= \big(a+c\big)+\big((-b-d)\big)$ | $A+(-B) =: A-B$ | $A=-b,\ B=d$ |
| $= \big(a+c\big)+\big(-(b+d)\big)$ | $-A-B = -(A+B)$ | $A=b,\ B=d$ |
| $= (a+c)-(b+d)$ | $A+(-B) =: A-B$ | $A=a+c,\ B=b+d$ |

| step | by | with |
|---|---|---|
| $(a/b)\cdot(cd) = \big(a\cdot b^{-1}\big)\cdot(c/d)$ | $A/B := A\cdot B^{-1}$ | $A=a,\ B=b$ |
| $\quad = \big(a\cdot b^{-1}\big) + \big(c\cdot d^{-1}\big)$ | $A/B := A\cdot B^{-1}$ | $A=c,\ B=d$ |
| $= \big(a\cdot c\big)\cdot\big(b^{-1}\cdot d^{-1}\big)$ | $(A\cdot B)\cdot(C\cdot D) =$ | $A=a,\ B=b^{-1},$ |
| $= \big(a\cdot c\big)\cdot\big(b^{-1}\cdot d^{-1}\big)$ | $\quad = (A\cdot C)\cdot(B\cdot D)$ | $C=c,\ D=d^{-1}$ |
| $\quad = \big(a\cdot c\big)\cdot(b^{-1}\cdot d^{-1})$ | $A\cdot B^{-1} =: A/B$ | $A=b^{-1},\ B=d$ |
| $\quad = \big(a\cdot c\big)\cdot(b\cdot d)^{-1}$ | $A^{-1}\cdot B^{-1} = (A\cdot B)^{-1}$ | $A=b,\ B=d$ |
| $\quad = (a\cdot c)/(b\cdot d)$ | $A\cdot B^{-1} =: A/B$ | $A=a\cdot c,\ B=b\cdot d$ |

□

Theorem A.6. $a-b=(a+c)-(b+c)$ and similarly $\dfrac{a}{b}=\dfrac{ac}{bc}$.

*Proof.*

| step | by | with |
|---|---|---|
| $a-b=(a-b)+0$ | $A=A+0$ | $A=a-b$ |
| $\quad = (a-b)-(c-c)$ | $0=A-A$ | $A=c$ |
| $\quad = (a+c)-(b+c)$ | $(A-B)+(C-D) =$ | $A=a,\ B=b,$ |
| | $\quad = (A+C)-(B+D)$ | $C=D=c$ |

| step | by | with |
|---|---|---|
| $a/b=(a/b)\cdot 1$ | $A=A\cdot 1$ | $A=a/b$ |
| $\quad = (a/b)/(c/c)$ | $1=A/A$ | $A=c$ |
| $\quad = (a\cdot c)/(b\cdot c)$ | $(A/B)\cdot(C/D) =$ | $A=a,\ B=b,$ |
| | $\quad = (A\cdot C)/(B\cdot D)$ | $C=D=c$ |

□

Theorem A.7. $(a-b)-(c-d)=(a-b)+(d-c)$ and similarly $\dfrac{a/b}{c/d}=\dfrac{a}{b}\cdot\dfrac{d}{c}$.

*Proof.*

| step | by | with |
|---|---|---|
| $(a-b)-(c-d) =$ | | |
| $\quad = \big(a-b\big)+\big(-(c-d)\big)$ | $A-B := A+(-B)$ | $A=a-b,\ B=c-d$ |
| $\quad = \big(a-b\big)+\big(d-c\big)$ | $-(A-B)=B-A$ | $A=c,\ B=d$ |

| step | by | with |
|---|---|---|
| $(a/b)/(c/d) = \big(a/b\big) \cdot \big(c/d\big)^{-1}$ | $A/B := A \cdot B^{-1}$ | $A = a/b$, $B = c/d$ |
| $\quad = \big(a/b\big) \cdot \big(d/c\big)$ | $(A/B)^{-1} = B/A$ | $A = c$, $B = d$ |

□

## A.3 CONSEQUENCES OF THE DISTRIBUTIVE LAW

| step | by | with |
|---|---|---|
| $(a+b)c = c(a+b)$ | $AB = BA$ | $A = (a+b)$, $B = c$ |
| $= ca + cb$ | $A(B+C) = AB + AC$ | $A = c$, $B = a$, $C = b$ |
| $= ac + cb$ | $AB = BA$ | $A = c$, $B = a$ |
| $= ac + bc$ | $AB = BA$ | $A = c$, $B = b$ |

THEOREM A.8. $a \cdot 0 = 0$

*Proof.*

| step | by | with |
|---|---|---|
| $0 = -a + a$ | $0 = -A + A$ | $A = a$ |
| $= -a + a \cdot 1$ | $A = A \cdot 1$ | $A = a$ |
| $= -a + a \cdot (1 + 0)$ | $A = A + 0$ | $A = 1$ |
| $= -a + (a \cdot 1 + a \cdot 0)$ | $A \cdot (B + C) = A \cdot B + A \cdot C$ | $A = a$, $B = a \cdot 1$, $C = 0$ |
| $= (-a + a \cdot 1) + a \cdot 0$ | $A + (B + C) = A \cdot (A + B) + C$ | $A = -a$, $B = a \cdot 1$, $C = a \cdot 0$ |
| $= (-a + a) + a \cdot 0$ | $A \cdot 1 = A$ | $A = a$ |
| $= 0 + a \cdot 0$ | $-A + A = 0$ | $A = a$ |
| $= a \cdot 0$ | $0 + A = A$ | $A = a \cdot 0$ |

□

THEOREM A.9. $-a = (-1)a$

*Proof.*

| step | by | with |
|---|---|---|
| $0 = 0 \cdot a$ | $0 = A \cdot 0 = 0 \cdot A$ | $A = a$ |
| $= (1 + (-1)) \cdot a$ | $A + (-A) = 0$ | $A = a$ |
| $= 1 \cdot a + (-1) \cdot a$ | $(A + B) \cdot C = A \cdot C + B \cdot C$ | $A = 1$, $B = -1$, $C = a$ |
| $= a + (-1) \cdot a$ | $1 \cdot A = A$ | $A = a$ |
| $(-1) \cdot a = -a$ | $A + B = 0 \implies B = -A$ | $B = (-1)a$, $A = a$ |

□

Theorem A.10. $a(-b) = -ab$

*Proof.*

| step | by | with |
|---|---|---|
| $a \cdot (-b) = a \cdot [(-1) \cdot b]$ | $-A = (-1)A$ | $A = b$ |
| $= [a \cdot (-1)] \cdot b$ | $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ | $A = a$, $B = -1$, $C = b$ |
| $= [(-1) \cdot a] \cdot b$ | $A \cdot B = B \cdot A$ | $A = a$, $B = -1$ |
| $= (-1) \cdot [a \cdot b]$ | $A \cdot B) \cdot C) = (A \cdot (B \cdot C$ | $A = -1$, $B = a$, $C = b$ |
| $-(a \cdot b)$ | $(-1)A = -A$ | $A = a \cdot b$ |

□

Theorem A.11. $(-a)(-b) = ab$

*Proof.*

| step | by | with |
|---|---|---|
| $(-a) \cdot (-b) = -[(-a) \cdot b]$ | $A \cdot (-B) = -(A \cdot B)$ | $A = -a$, $B = b$ |
| $= -[b \cdot (-a)]$ | $A \cdot B = B \cdot A$ | $A = -a$, $B = b$ |
| $= -[-(b \cdot a)]$ | $A \cdot (-B) = -(A \cdot B)$ | $A = b$, $B = a$ |
| $= -[-(a \cdot b)]$ | $A \cdot B = B \cdot A$ | $A = b$, $B = a$ |
| $= a \cdot b$ | $-(-A) = A$ | $A = a \cdot b$ |

□

Theorem A.12. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + cb}{bd}$

*Proof.*

| step | by | with |
|---|---|---|
| $(a+b)(c+d) = (a+b)c+(a+b)d$ | $(A(B+C) = AB + AC$ | $A = (a+b)$, $B = c$, $C = d$ |
| $= ac + bc + (a+b)d$ | $(A+B)C = AC + BC$ | $A = a$, $B = b$, $C = c$ |
| $= ac + bc + ad + bd$ | $(A+B)C = AC + BC$ | $A = a$, $B = b$, $C = d$ |

□

THEOREM A.13. $(a+b)^2 = a^2 + 2ab + b^2$

*Proof.*

| step | by | with |
|---|---|---|
| $(a+b)^2 = (a+b)(a+b)$ | $A^2 = A \cdot A$ | $A = (a+b)$ |
| $= a^2 + ab + ba + b^2$ | $A+B)(C+D) = AB+AD+BC+CD$ | $A = C = a$, $B = D = b$ |
| $= a^2 + ab + ab + b^2$ | $AB = BA$ | $A = a$, $B = b$ |
| $= a^2 + 1 \cdot ab + 1 \cdot ab + b^2$ | $A = 1 \cdot A$ | $A = ab$ |
| $= a^2 + (1+1)ab + b^2$ | $AC + BC = (A+B)C$ | $A = B = 1$, $C = ab$ |
| $= a^2 + 2ab + b^2$ | $1 + 1 = 2$ | $2 := 1 + 1$ |

□

THEOREM A.14. $(a+b)(a-b) = a^2 - b^2$

*Proof.*

| step | by | with |
|---|---|---|
| $(a-b)(a+b) = a^2 + ab + (-b)a + (-b)b$ | $(A+B)(C+D) = AC+AD+BC+BD$ | $A = C = a$, $B = -b$, $D = b$ |
| $(a-b)(a+b) = a^2 + ab + (-b)a - b^2$ | $(-A)A = -a^2$ | $A = b$ |
| $= a^2 + ab + a(-b) - b^2$ | $AB = BA$ | $A = -b$, $B = a$ |
| $= a^2 + ab - ab - b^2$ | $A(-B) = -AB$ | $A = a$, $B = b$ |
| $= a^2 + 0 - b^2$ | $A - A = 0$ | $A = ab$ |
| $= a^2 - b^2$ | $A + 0 = A$ | $A = a^2$ |

□